Vector spaces
(aka *linear* spaces)

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Vector space: definition & addition operation

A vector/linear space is a set of elements, named *vectors*, with the following properties:

1. There exists an operation called addition, represented by the symbol $+$ such that $x + y \in V$ for every $x, y \in V$, i.e.,

$$x, y \in V \Longrightarrow x + y \in V$$

   This operation must satisfy the following **properties**:

   a) Commutativity: $\forall x, y \in V, x + y = y + x$
   b) Associativity: $\forall x, y, z \in V, x + (y + z) = (x + y) + z$
   c) Identity element: $\exists\ 0 \in V \mid \forall x \in V, x + 0 = 0 + x = x$
   d) Inverse element: $\forall x \in V\ \exists(-x) \mid x + (-x) = 0$

Vector space: scalar multiplication operation

2. There exists an operation called scalar multiplication that takes a scalar $\alpha \in F$ (with F being a *field*) and a vector $x \in V$ to yield another vector, i.e,

$$\forall x \in V,\ \forall \alpha \in F \implies \alpha x \in V$$

This operation must satisfy the following **properties**:

a) Associativity: $\forall \alpha, \beta \in F, \forall x \in V, \alpha(\beta x) = (\alpha\beta)x$
b) Identity element: $\exists\ 1 \in F \mid \forall x \in V, 1x = x$
c) Distributivity of scalar multiplication with respect to vector addition:

$$\forall \alpha \in F, \forall x, y \in V, \alpha(x + y) = \alpha x + \alpha y$$

d) Distributivity of scalar multiplication with respect to field addition:

$$\forall \alpha, \beta \in F, \forall x \in V, (\alpha + \beta)x = \alpha x + \beta x$$