Stochastic processes

Introduction

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Stochastic processes

- Extension of the concept of random variable that accounts for temporal dependence
  - Random variable

$$\omega \in \Omega \to X(\omega)$$

  - Stochastic process

$$\omega \in \Omega \to X(t, \omega)$$

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- Specifics
  - $X(t_i, \omega_j)$: individual realization
  - $X(t, \omega_i)$: time-varying signal associated with ω_i, $x_i(t)$
  - $X(t_i, \omega)$: random variable ($X(\omega)$)

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- Notation: $X(t)$ or $X[n]$

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- Notation: $X(t)$ or $X[n]$
- Interpretation: Indexed set of random variables
  - Continuous index ($t \in \mathbb{R}$): Continuous random process
  - Discrete index ($n \in \mathbb{Z}$): Discrete random process

Description of a stochastic process

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- Analytical description

$$X(t) = f(t, \boldsymbol{\theta})$$

with $\boldsymbol{\theta} = \{\theta_1, \theta_2, \cdots, \theta_n\}$ being a vector of random variables.
It entails knowing:

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- Statistical description
  - Complete (statistical description): $\forall\ (t_1, t_2, \cdots, t_n) \in \mathbb{R}^n$, $\forall\ n$

$$f_{X(t_1), X(t_2), \cdots, X(t_n)}(x_1, x_2, \cdots, x_n)$$

  - M-th order statistics: $\forall\ n \leq M$, $\forall\ (t_1, t_2, \cdots, t_n) \in \mathbb{R}^n$

$$f_{X(t_1), X(t_2), \cdots, X(t_n)}(x_1, x_2, \cdots, x_n)$$

Expectations of the process (statistical averages)

- Mean of a process

$$\mu_X(t) = \mathbb{E}\left[X(t)\right] = \int_{-\infty}^{\infty} x\, f_{X(t)}(x) dx$$

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$$R_X(t_1, t_2) = \mathbb{E}\left[X(t_1)X(t_2)^*\right]$$

$$R_X(t_1, t_2) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x_1\, x_2 f_{X(t_1),X(t_2)^*}(x_1, x_2)\cdot dx_1 \cdot dx_2$$

Notation

Very often we write $R_X(t_1, t_2)$ as $R_X(t+\tau, t)$ with $t = t_2$ and $\tau = t_1 - t_2$

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- Autocovariance of a process

$$\begin{aligned}\mathrm{Cov}_X(t_1, t_2) &= \mathbb{E}\left[(X(t_1) - \mu_X(t_1))\,(X(t_2) - \mu_X(t_2))^*\right] \\ &= R_X(t_1, t_2) - \mu_X(t_1)\mu_X(t_2)\end{aligned}$$

Example of stochastic process I

- Definition of the random experiment
  - Four possible results: ω_1, ω_2, ω_3, ω_4
  - Probabilities: $P(\omega_1) = P(\omega_2) = P(\omega_3) = P(\omega_4) = \frac{1}{4}$
- Analytical description of the process

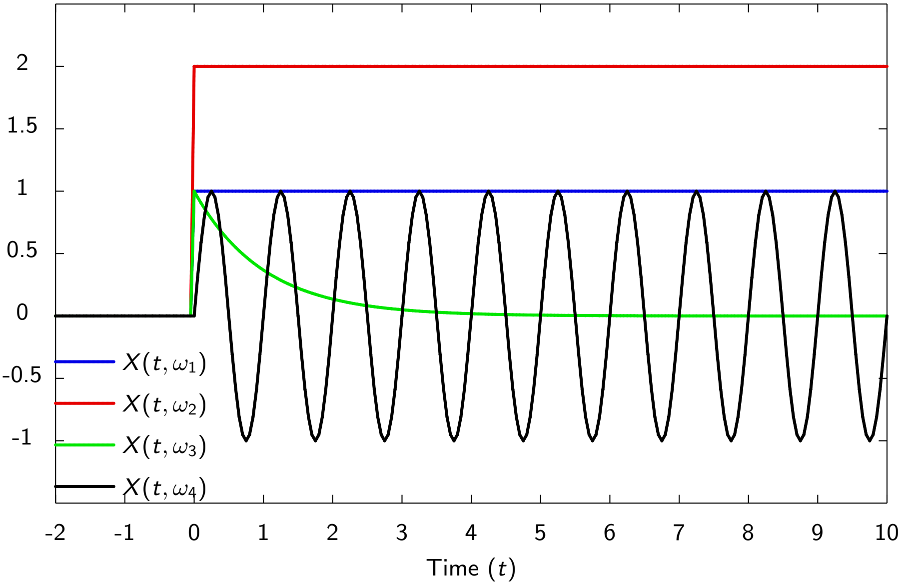
$$X(t,\omega_1) = \begin{cases} 1, & \text{if } t \geq 0 \\ 0, & \text{if } t < 0 \end{cases}$$

$$X(t,\omega_2) = \begin{cases} 2, & \text{if } t \geq 0 \\ 0, & \text{if } t < 0 \end{cases}$$

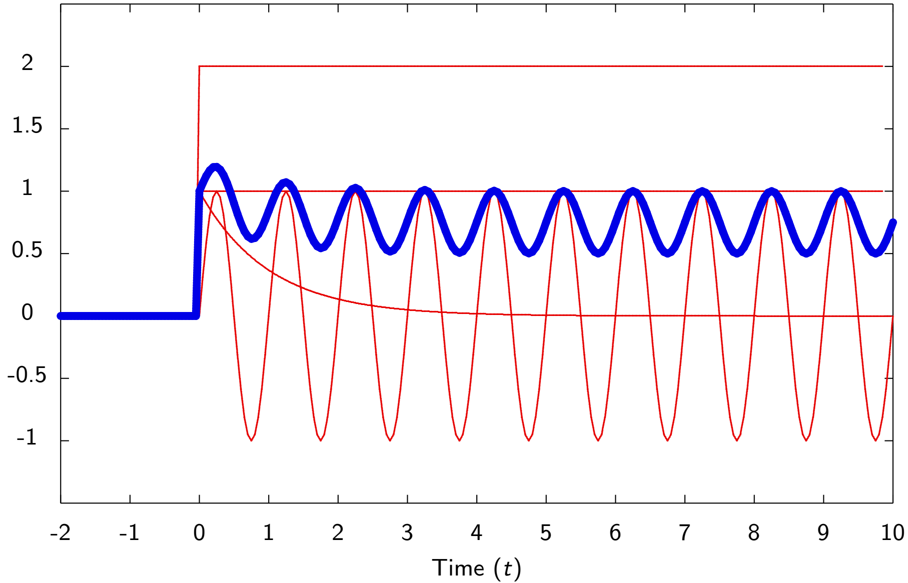
$$X(t,\omega_3) = \begin{cases} e^{-t}, & \text{if } t \geq 0 \\ 0, & \text{if } t < 0 \end{cases}$$

$$X(t,\omega_4) = \begin{cases} \sin(t), & \text{if } t \geq 0 \\ 0, & \text{if } t < 0 \end{cases}$$

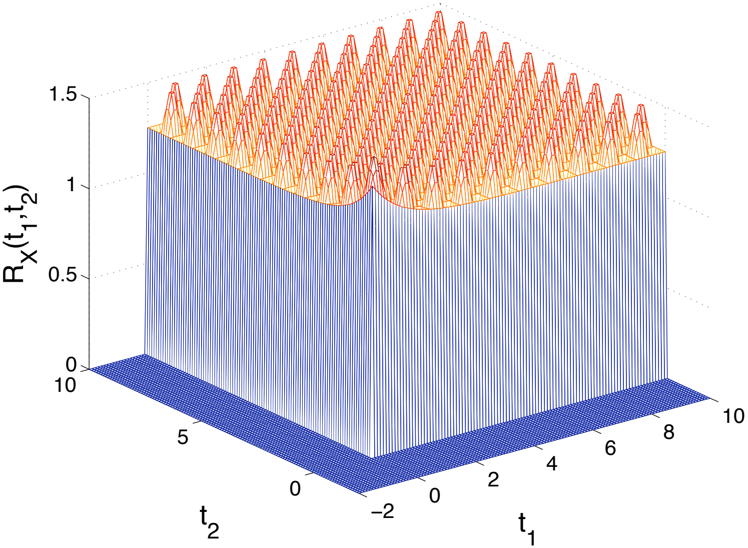
Signals from the random process I



Mean of the random process I



Autocorrelation of the random process I



Example of stochastic process II

- Gaussian random process

$$f_{X(t_1),X(t_2),\cdots,X(t_n)}(x_1, x_2, \cdots, x_n) = \frac{1}{\sqrt{(2\pi)^n \det(\boldsymbol{C})}} \times$$
$$\times\, e^{-\frac{1}{2}(\mathbf{x}-\boldsymbol{\mu})\boldsymbol{C}^{-1}(\mathbf{x}-\boldsymbol{\mu})^T}$$

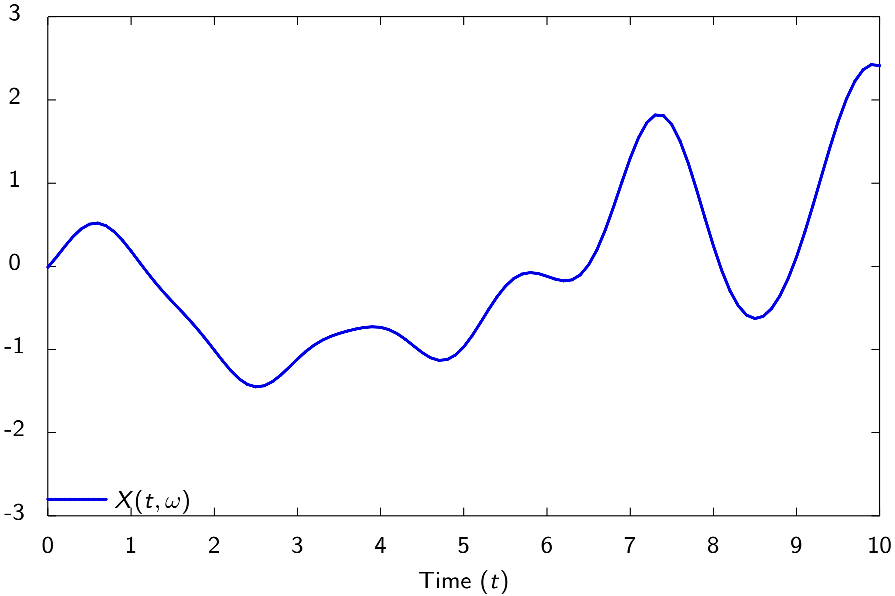
  - Vector of means: $\boldsymbol{\mu} = [\mu_1, \mu_2, \cdots, \mu_n]^T$

$$\mu_i = E[X(t_i)] = 0$$

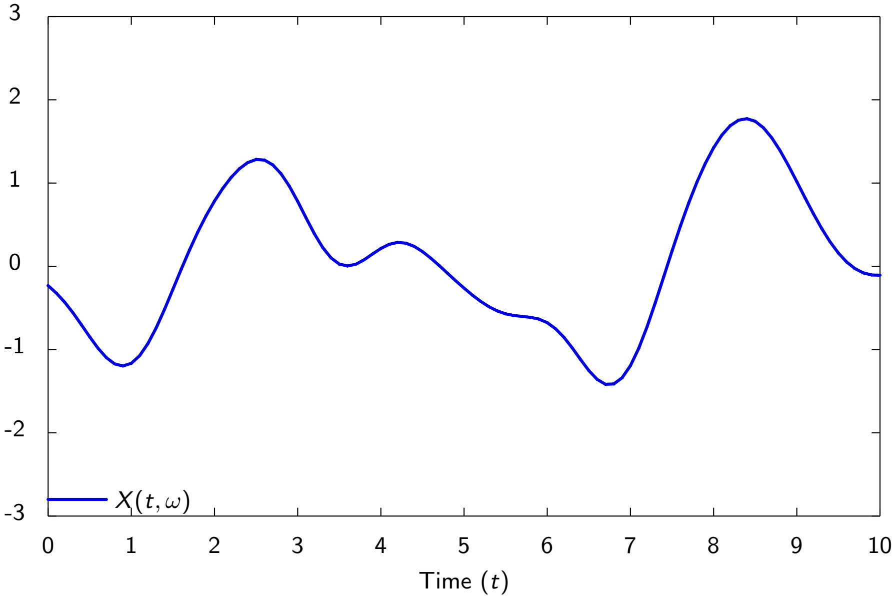
  - Matrix of covariances: $\boldsymbol{C}$, given by

$$C_{i,j} = \mathrm{Cov}(X(t_i), X(t_j)) = e^{|t_i - t_j|^2}$$

Realization of the random process II



Another realization of the random process II



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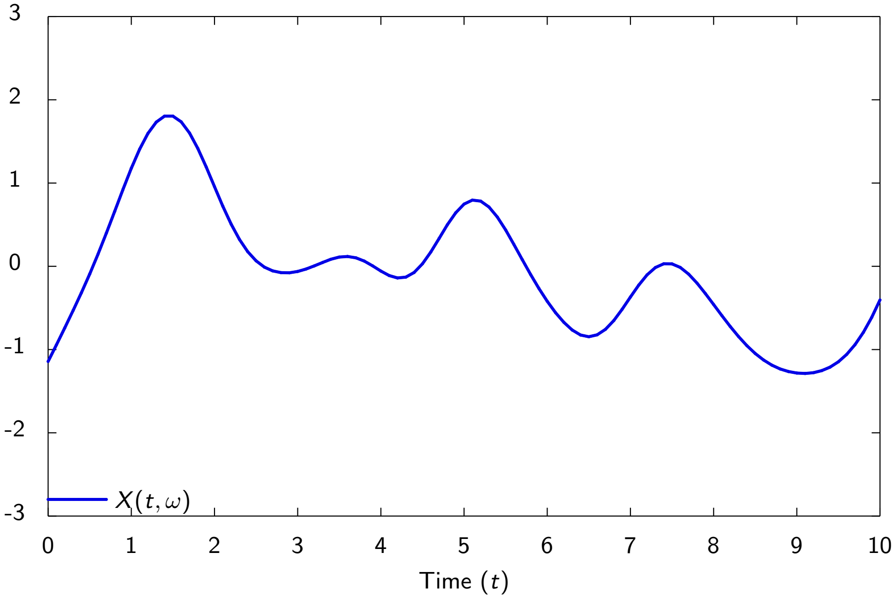

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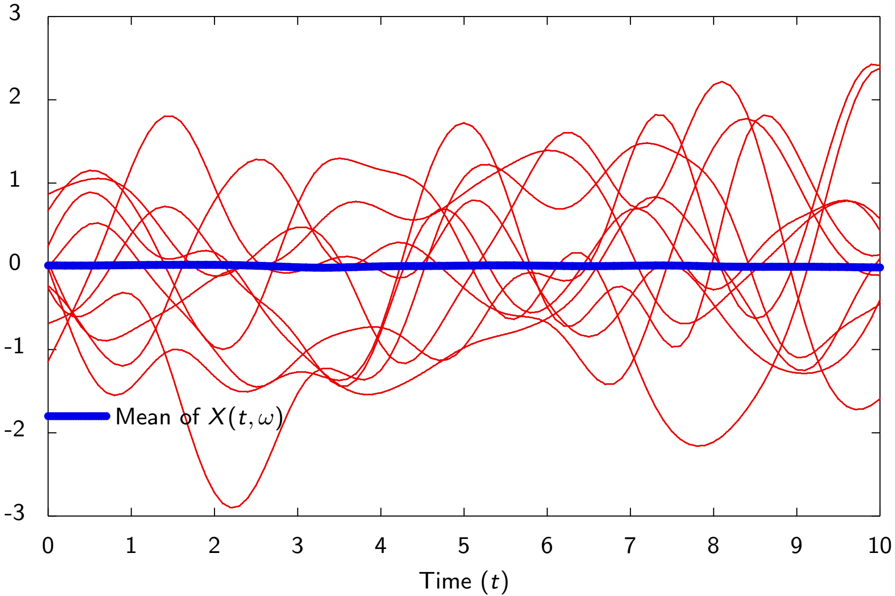
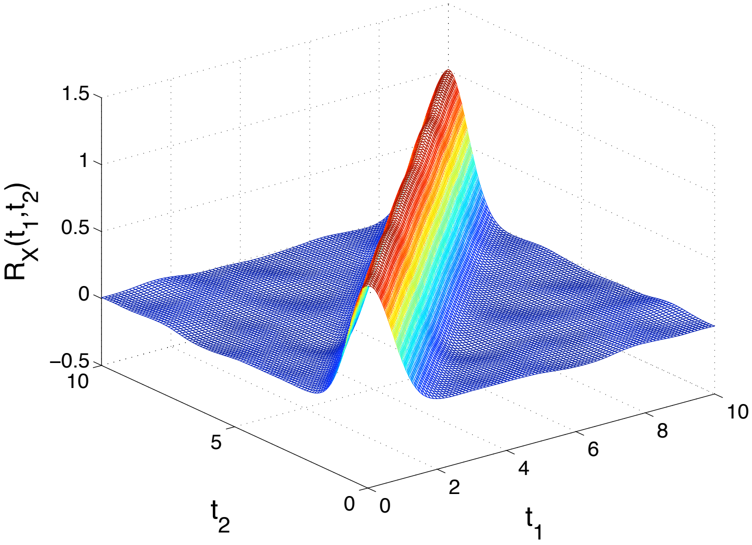


Autocorrelation of the random process II



Stationarity

- Strict stationarity: $\forall\ (t_1, t_2, \cdots, t_n)$, $\forall n$, $\forall \Delta$

$$f_{X(t_1),X(t_2),\cdots,X(t_n)}(x_1, x_2, \cdots, x_n) = f_{X(t_1+\Delta),X(t_2+\Delta),\cdots,X(t_n+\Delta)}(x_1, x_2, \cdots, x_n)$$

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- Wide-sense stationarity
  1. $\mu_X(t) = \mu_X$ (it does not depend on t) $\Rightarrow$ **mean-stationary**
  2. $R_X(t_1, t_2) = R_X(t_1 - t_2) = R_X(\tau) \Rightarrow$ **autocorrelation-stationary** with $\tau = t_1 - t_2$.

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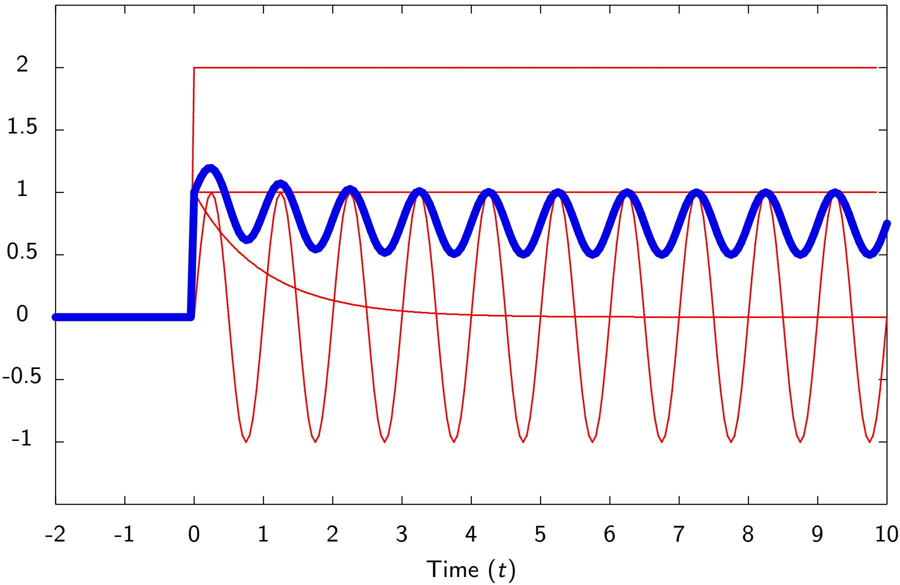
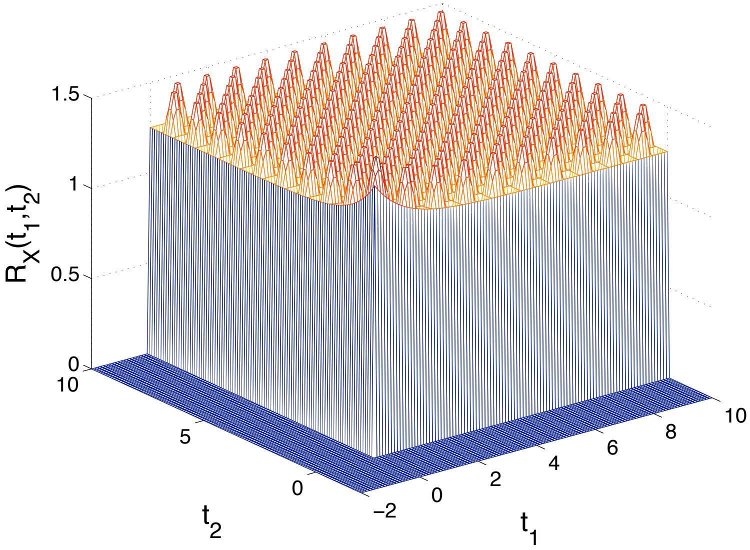


Autocorrelation of the random process I



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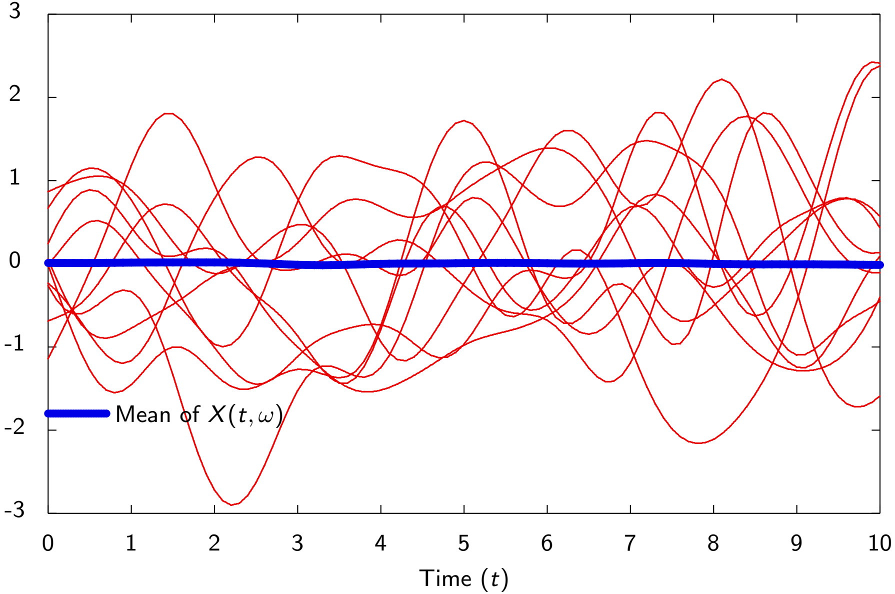

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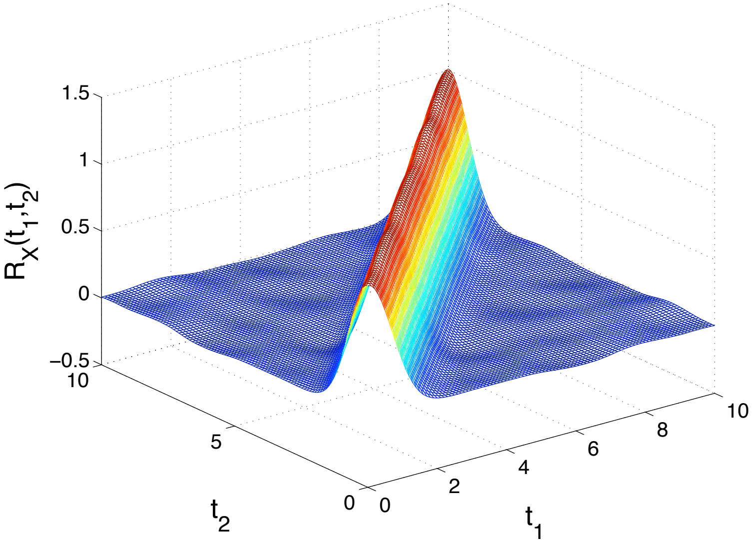

Properties of the autocorrelation of WSS processes

The autocorrelation function, $R_X(\tau)$, of a wide-sense stationary process $X(t)$, has the following properties:

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$$R_X(-\tau) = R_X(\tau)$$

- The modulus is maximum a the origin $\tau = 0$

$$|R_X(\tau)| \leq R_X(0)$$

Expectations of two processes

- Cross-correlation between $X(t)$ and $Y(t)$

$$R_{XY}(t_1, t_2) = \mathbb{E}\left[X(t_1) \cdot Y(t_2)^*\right]$$
$$R_{YX}(t_2, t_1) = R_{XY}^*(t_1, t_2)$$

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- Cross-covariance between X and Y

$$\mathrm{Cov}_{XY}(t_1, t_2) = \mathbb{E}\left[(X(t_1) - \mu_X(t_1))\,(Y(t_2) - \mu_Y(t_2))^*\right]$$
$$\mathrm{Cov}_{YX}(t_1, t_2) = \mathrm{Cov}_{XY}^*(t_2, t_1)$$

Joint stationarity

Two processes, $X(t)$ and $Y(t)$ are wide-sense jointly stationary if

- they both are individually WSS
- the cross-correlation between them only depends on the time difference, i.e.,

  $$R_{XY}(t_1, t_2) = R_{XY}(\tau)$$

  with $\tau = t_1 - t_2$