Fundamental limits in communications

Communications Theory

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Performance of a communications system

How to measure the performance of a system? using...

- ...the probability of error, P_e
  - $\uparrow$ distance between elements in the constellation $\Rightarrow \downarrow P_e$
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- $\uparrow$ information lost $\Rightarrow \downarrow$ performance

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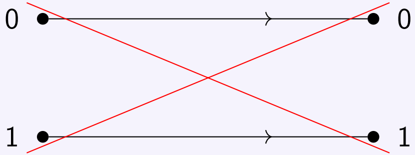
a random connection!! not deterministic

$X, Y \in \{0, 1\}$



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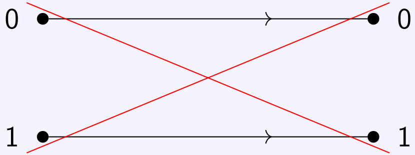

- a **quantitative** measure of information: how much information is lost from the input to the output

(Probabilistic) Channel Models

They model the connection between the transmitted and received symbols,

$$X \equiv \text{input}$$
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$$\textit{Probabilistic}\ \text{channel model} \xrightarrow{\text{yields}} f_{Y|X}(y|x)$$

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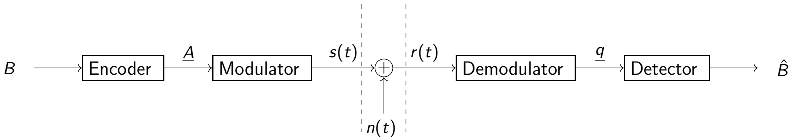

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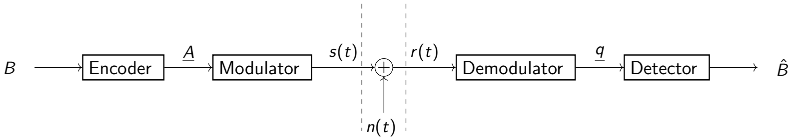
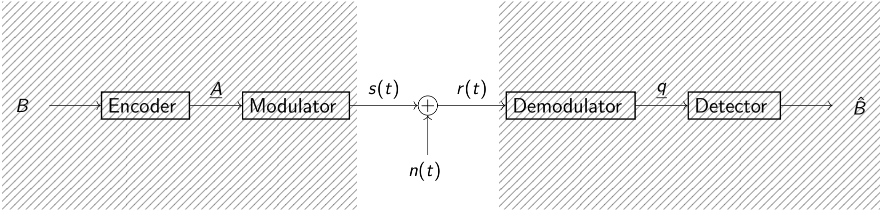
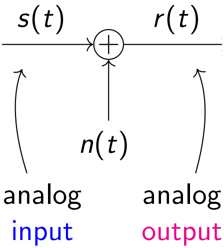


,and depending on what we consider the input and output, we have **different channel models**...

Gaussian channel





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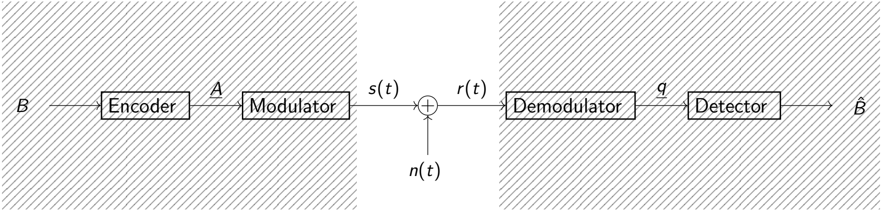
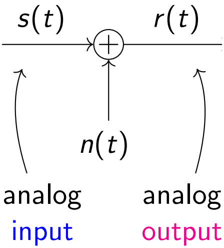
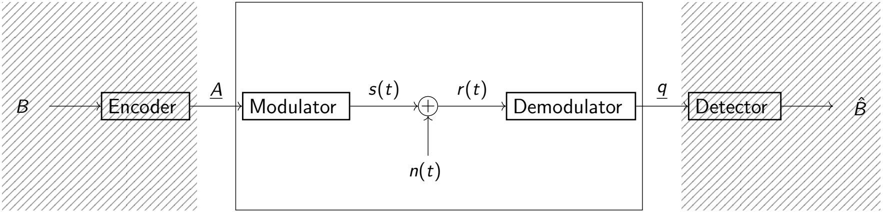
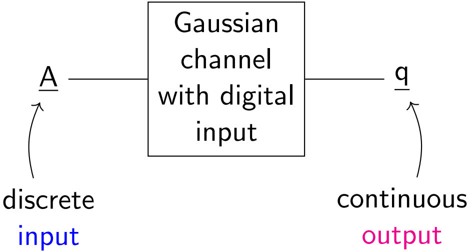


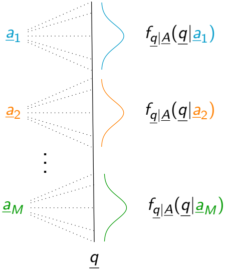


The model is specified by the pdf of $r(t)|s(t) \sim \mathcal{N}\left(s(t), \sigma_n^2\right)$

Gaussian channel with digital input







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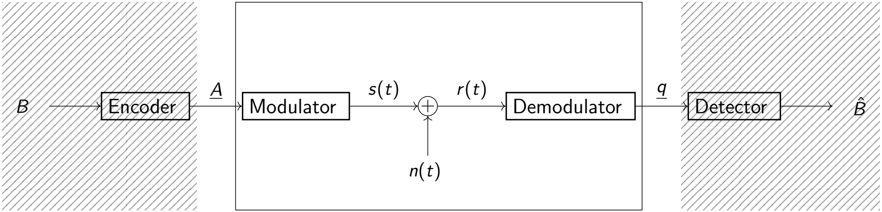
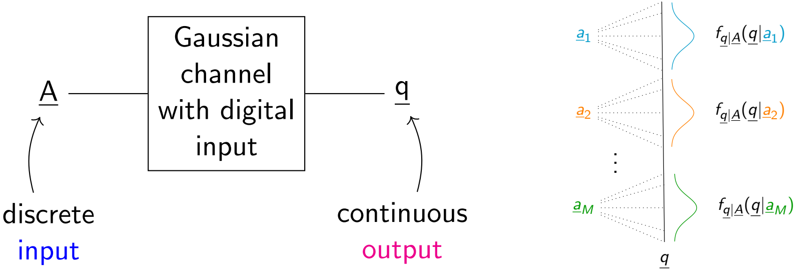
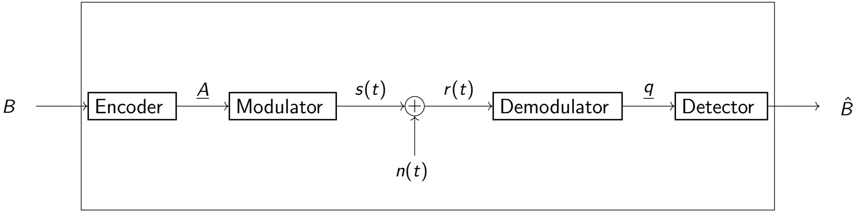
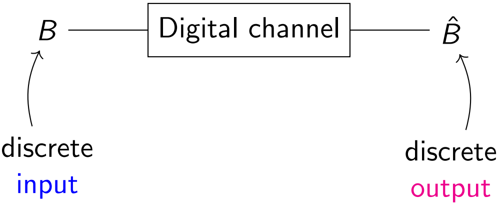




...known as the **discrete-time equivalent channel**. The model is specified by the pdfs $f_{\underline{q}|\underline{A}}(\underline{q}|\underline{a}_i), i = 1, \cdots, M$

Digital channel





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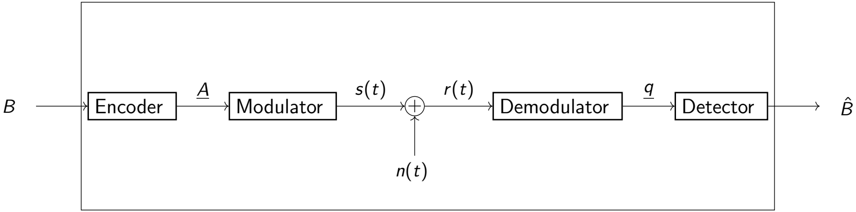

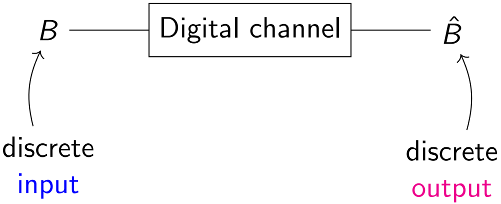

- input and output alphabets are **the same**

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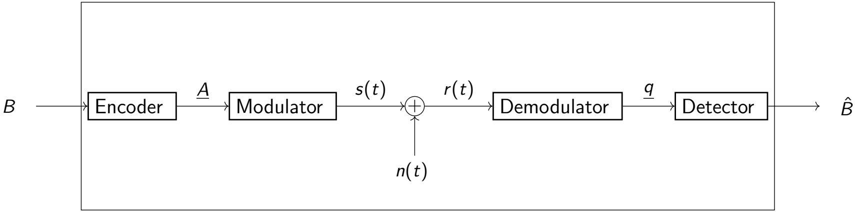
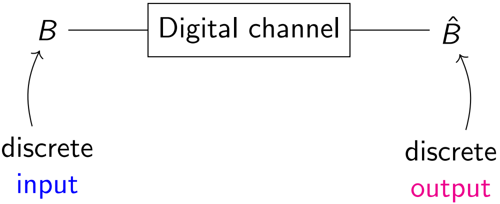
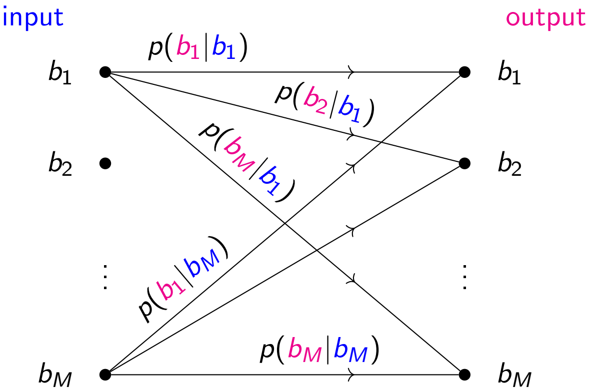




- input and output alphabets are **the same**
- model is specified by the transition probabilities $p(b_j|b_i), i, j = 1, \cdots, M$

Trellis representation of a *digital channel*

$$B, \hat{B} \in \{b_1, b_2, \cdots, b_M\}$$



$p(b_j|b_i) \equiv$ probability of receiving b_j when b_i was transmitted

Discrete memoryless channel (DMC)

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The DMC is a *generalization* of the previous model in which the input and output alphabets can be different.

$$X \longrightarrow \boxed{\text{DMC}} \longrightarrow Y$$

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being

- $X \in \underbrace{\{x_1, x_2, \cdots, x_M\}}_{\text{input alphabet}}$ is a random variable
- $Y \in \underbrace{\{y_1, y_2, \cdots, y_L\}}_{\text{output alphabet}}$ is a *different* random variable

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Transition probability matrix (channel matrix)

$$\underline{\underline{P}} = \begin{bmatrix} p(y_1|x_1) & p(y_2|x_1) & \cdots & p(y_L|x_1) \\ p(y_1|x_2) & \ddots & \cdots & p(y_L|x_2) \\ \vdots & \ddots & \ddots & \vdots \\ p(y_1|x_M) & p(y_2|x_M) & \cdots & p(y_L|x_M) \end{bmatrix}$$

- rows add up to 1 (i-th row is the pmf of Y conditional on x_i)
- columns do **not** add up to 1

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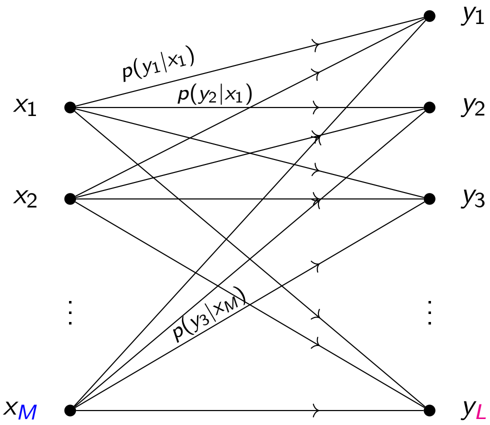

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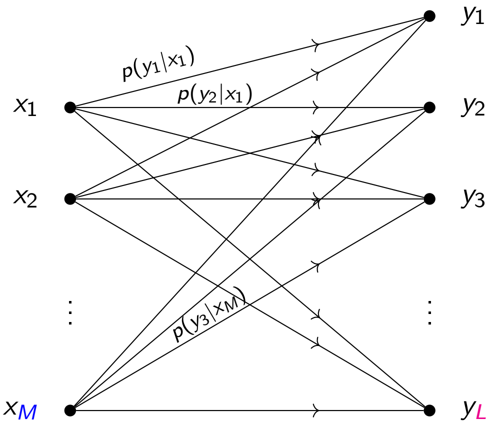

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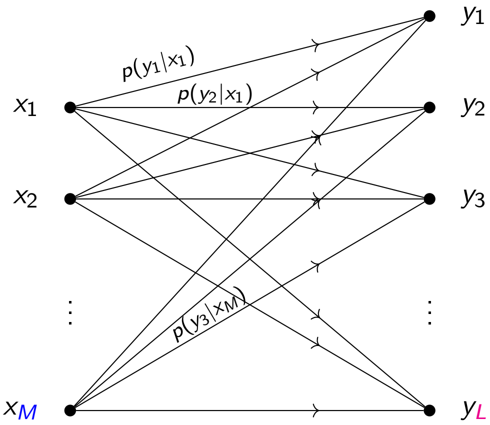

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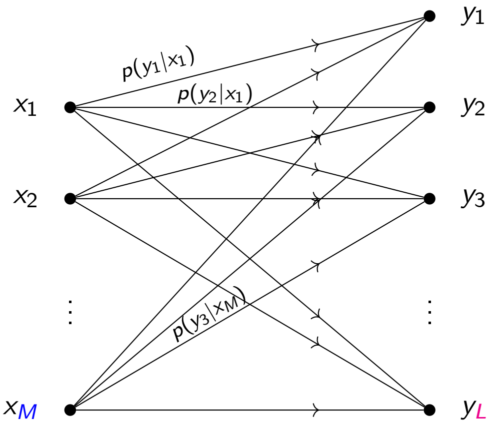

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- $\sum_{i=1}^{M} p(x_i) = 1$, $\sum_{i=1}^{L} p(y_i) = 1$
- $p(y_j) = \sum_{i=1}^{M} p(x_i, y_j) = \sum_{i=1}^{M} p(y_j|x_i)p(x_i)$

Example: computation of the transition probabilities

$M = 4$, equally likely symbols $\left(p(a_i) = \frac{1}{4}\right)$, Gaussian noise with $S_n(j\omega) = \frac{N_0}{2}$

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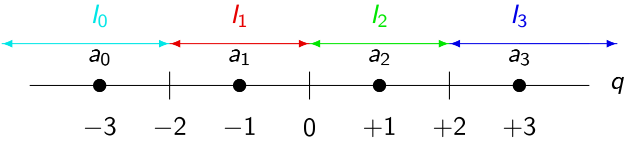
- Decision regions: thresholds $q_t = -2,\ q_t' = 0,\ q_t'' = +2$

$$I_0 = (-\infty, -2],\ I_1 = (-2, 0],\ I_2 = (0, +2],\ I_3 = (+2, +\infty)$$



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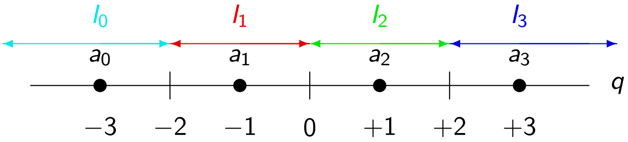

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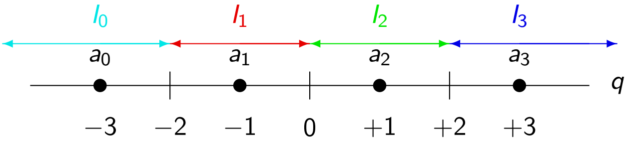
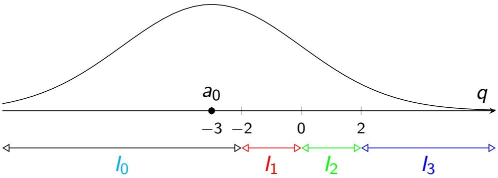


In this case: $x_0 = y_0 = a_0, \cdots, x_M = y_M = a_M \ldots$ and hence the transition probability matrix (channel matrix) is given by

$$\underline{\underline{P}} = \begin{bmatrix} p(a_0 \mid a_0) & p(a_1 \mid a_0) & p(a_2 \mid a_0) & p(a_3 \mid a_0) \\ p(a_0 \mid a_1) & p(a_1 \mid a_1) & p(a_2 \mid a_1) & p(a_3 \mid a_1) \\ p(a_0 \mid a_2) & p(a_1 \mid a_2) & p(a_2 \mid a_2) & p(a_3 \mid a_2) \\ p(a_0 \mid a_3) & p(a_1 \mid a_3) & p(a_2 \mid a_3) & p(a_3 \mid a_3) \end{bmatrix}$$

Example: elements in the first row: $p_{Y|X}(y_j|x_0)$, $\forall j$



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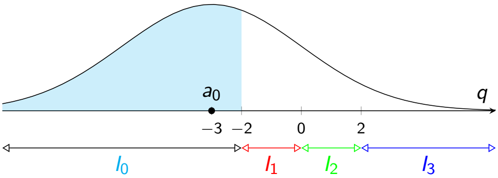

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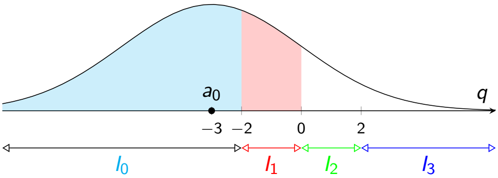

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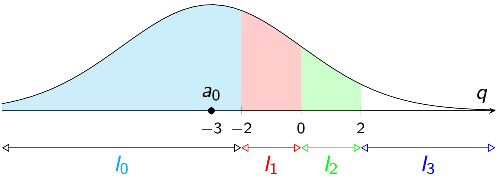

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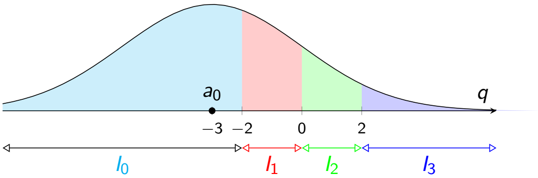

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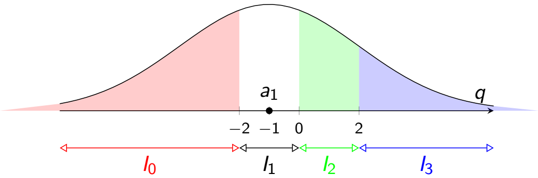


- $f_{q|a}(q|a_1)$ distribution: Gaussian with mean a_1 and variance $N_0/2$

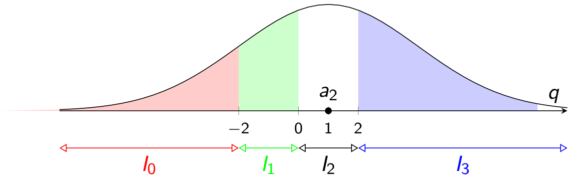
$$p_{Y|X}(a_0|a_1) = P_{e|a_1 \to a_0} = \mathrm{Q}\left(\frac{1}{\sqrt{N_0/2}}\right)$$

$$p_{Y|X}(a_1|a_1) = 1 - P_{e|a_1} = 1 - 2\mathrm{Q}\left(\frac{1}{\sqrt{N_0/2}}\right)$$

$$p_{Y|X}(a_2|a_1) = P_{e|a_1 \to a_2} = \mathrm{Q}\left(\frac{1}{\sqrt{N_0/2}}\right) - \mathrm{Q}\left(\frac{3}{\sqrt{N_0/2}}\right)$$

$$p_{Y|X}(a_3|a_1) = P_{e|a_1 \to a_3} = \mathrm{Q}\left(\frac{3}{\sqrt{N_0/2}}\right)$$

Example: elements in the third row: $p_{Y|X}(y_j|a_2)$, $\forall j$



- $f_{q|a}(q|a_2)$ distribution: Gaussian with mean a_2 and variance $N_0/2$

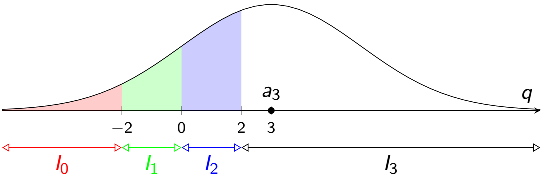
$$p_{Y|X}(a_0|a_2) = P_{e|a_2 \to a_0} = \mathrm{Q}\left(\frac{3}{\sqrt{N_0/2}}\right)$$

$$p_{Y|X}(a_1|a_2) = P_{e|a_2 \to a_1} = \mathrm{Q}\left(\frac{1}{\sqrt{N_0/2}}\right) - \mathrm{Q}\left(\frac{3}{\sqrt{N_0/2}}\right)$$

$$p_{Y|X}(a_2|a_2) = 1 - P_{e|a_2} = 1 - 2\mathrm{Q}\left(\frac{1}{\sqrt{N_0/2}}\right)$$

$$p_{Y|X}(a_3|a_2) = P_{e|a_2 \to a_3} = \mathrm{Q}\left(\frac{1}{\sqrt{N_0/2}}\right)$$

Example: elements in the fourth row: $p_{Y|X}(y_j|a_3), \forall j$



- $f_{q|a}(q|a_3)$ distribution: Gaussian with mean a_3 and variance $N_0/2$

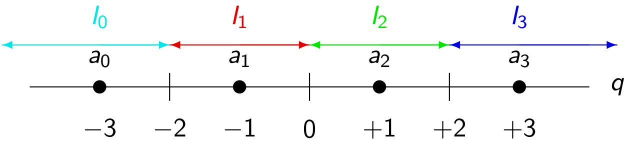
$$p_{Y|X}(a_0|a_3) = P_{e|a_3 \to a_0} = \mathrm{Q}\left(\frac{5}{\sqrt{N_0/2}}\right)$$

$$p_{Y|X}(a_1|a_3) = P_{e|a_3 \to a_1} = \mathrm{Q}\left(\frac{3}{\sqrt{N_0/2}}\right) - \mathrm{Q}\left(\frac{5}{\sqrt{N_0/2}}\right)$$

$$p_{Y|X}(a_2|a_3) = P_{e|a_3 \to a_2} = \mathrm{Q}\left(\frac{1}{\sqrt{N_0/2}}\right) - \mathrm{Q}\left(\frac{3}{\sqrt{N_0/2}}\right)$$

$$p_{Y|X}(a_3|a_3) = 1 - P_{e|a_3} = 1 - \mathrm{Q}\left(\frac{1}{\sqrt{N_0/2}}\right)$$

Example: wrap-up



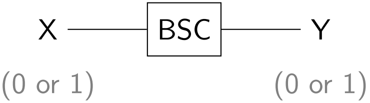
Channel matrix $\underline{\underline{P}}$ just collects together all the above probabilities:

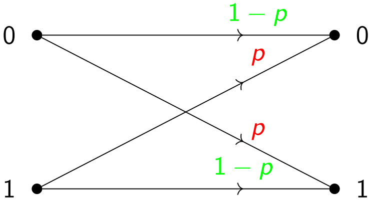
$$\underline{\underline{P}} = \begin{bmatrix} p(a_0 \mid a_0) & p(a_1 \mid a_0) & p(a_2 \mid a_0) & p(a_3 \mid a_0) \\ p(a_0 \mid a_1) & p(a_1 \mid a_1) & p(a_2 \mid a_1) & p(a_3 \mid a_1) \\ p(a_0 \mid a_2) & p(a_1 \mid a_2) & p(a_2 \mid a_2) & p(a_3 \mid a_2) \\ p(a_0 \mid a_3) & p(a_1 \mid a_3) & p(a_2 \mid a_3) & p(a_3 \mid a_3) \end{bmatrix}$$

$$= \begin{bmatrix} 1 - Q\left(\frac{1}{\sqrt{N_o/2}}\right) & Q\left(\frac{1}{\sqrt{N_o/2}}\right) - Q\left(\frac{3}{\sqrt{N_o/2}}\right) & Q\left(\frac{3}{\sqrt{N_o/2}}\right) - Q\left(\frac{5}{\sqrt{N_o/2}}\right) & Q\left(\frac{5}{\sqrt{N_o/2}}\right) \\ Q\left(\frac{1}{\sqrt{N_o/2}}\right) & 1 - 2Q\left(\frac{1}{\sqrt{N_o/2}}\right) & Q\left(\frac{1}{\sqrt{N_o/2}}\right) - Q\left(\frac{3}{\sqrt{N_o/2}}\right) & Q\left(\frac{3}{\sqrt{N_o/2}}\right) \\ Q\left(\frac{3}{\sqrt{N_o/2}}\right) & Q\left(\frac{1}{\sqrt{N_o/2}}\right) - Q\left(\frac{3}{\sqrt{N_o/2}}\right) & 1 - 2Q\left(\frac{1}{\sqrt{N_o/2}}\right) & Q\left(\frac{1}{\sqrt{N_o/2}}\right) \\ Q\left(\frac{5}{\sqrt{N_o/2}}\right) & Q\left(\frac{3}{\sqrt{N_o/2}}\right) - Q\left(\frac{5}{\sqrt{N_o/2}}\right) & Q\left(\frac{1}{\sqrt{N_o/2}}\right) - Q\left(\frac{3}{\sqrt{N_o/2}}\right) & 1 - Q\left(\frac{1}{\sqrt{N_o/2}}\right) \end{bmatrix}$$

Binary symmetric channel (BSC)

Particular case of DMC with $M = L = 2$





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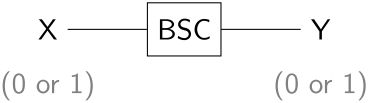

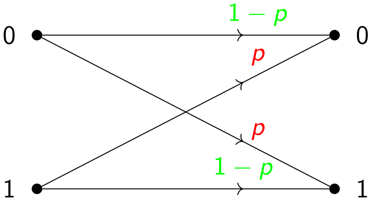

The labels of the trellis' edges yield the conditional probabilities:

$$\left.\begin{array}{l} p(1|0) = p(0|1) = p \equiv \text{ probability of error} \\ p(0|0) = p(1|1) = 1 - p \end{array}\right\} \Rightarrow \underline{\underline{P}} = \begin{bmatrix} 1-p & p \\ p & 1-p \end{bmatrix}$$

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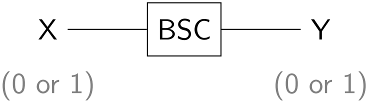

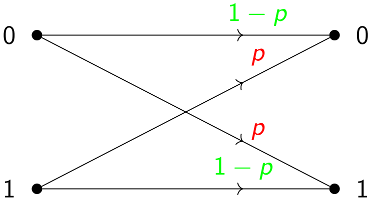

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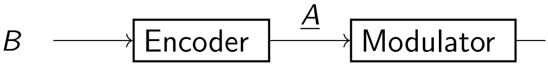
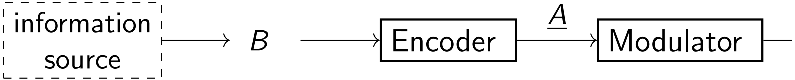
We can compute...

$$p(Y = 0) = p(0|0)p(0) + p(0|1)p(1) = (1 - p)p(0) + pp(1)$$
$$p(Y = 1) = p(1|0)p(0) + p(1|1)p(1) = pp(0) + (1 - p)p(1)$$

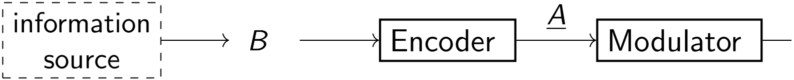
Modeling sources of information



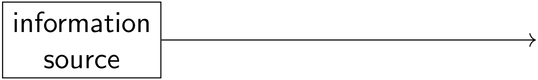
Modeling sources of information



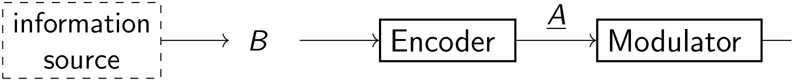
Modeling sources of information



We focus on **discrete-time** sources of information



Modeling sources of information



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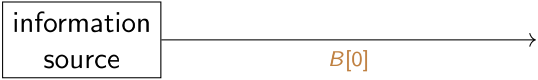

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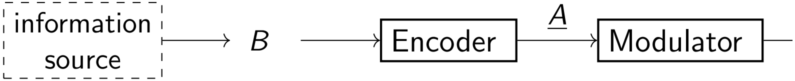

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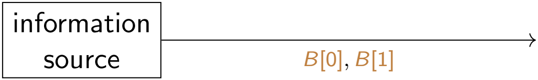

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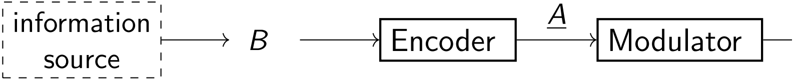

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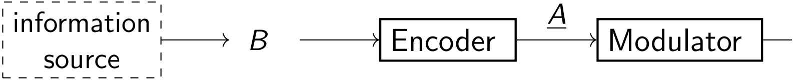

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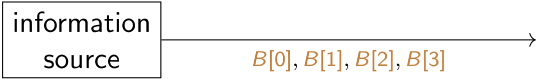

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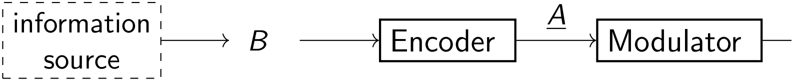

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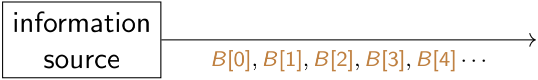

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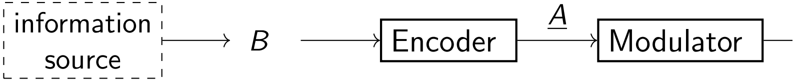

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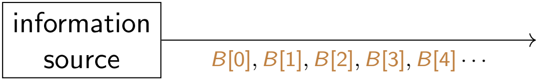

Every $B[i]$...

- ...will be a different B transmitted in our communications system
- ...is unknown $\Rightarrow$ it can be interpreted as a random variable $\Rightarrow$ an information source can be modelled as a collection of random variables $\{B[i]\}_{i=-\infty}^{\infty}$, i.e., a random process

For us, the $B[i]$'s are *independent and identically distributed*.