



Entropy and mutual information

Communication Theory

Manuel A. Vázquez

April 8, 2024

Joint entropy

If X, Y are **independent**

$$H(X, Y) = \sum_{i=1}^M \sum_{j=1}^L p(x_i, y_j) \log_2 \frac{1}{p(x_i, y_j)}$$

Joint entropy

If X, Y are **independent**

$$H(X, Y) = \sum_{i=1}^M \sum_{j=1}^L p(x_i, y_j) \log_2 \frac{1}{p(x_i, y_j)} = \sum_{i=1}^M \sum_{j=1}^L p(x_i)p(y_j) \log_2 \frac{1}{p(x_i)p(y_j)}$$

Joint entropy

If X, Y are **independent**

$$\begin{aligned} H(X, Y) &= \sum_{i=1}^M \sum_{j=1}^L p(x_i, y_j) \log_2 \frac{1}{p(x_i, y_j)} = \sum_{i=1}^M \sum_{j=1}^L p(x_i)p(y_j) \log_2 \frac{1}{p(x_i)p(y_j)} \\ &= \sum_{i=1}^M \sum_{j=1}^L p(x_i)p(y_j) \left(\log_2 \frac{1}{p(x_i)} + \log_2 \frac{1}{p(y_j)} \right) \end{aligned}$$

Joint entropy

If X, Y are **independent**

$$\begin{aligned} H(X, Y) &= \sum_{i=1}^M \sum_{j=1}^L p(x_i, y_j) \log_2 \frac{1}{p(x_i, y_j)} = \sum_{i=1}^M \sum_{j=1}^L p(x_i)p(y_j) \log_2 \frac{1}{p(x_i)p(y_j)} \\ &= \sum_{i=1}^M \sum_{j=1}^L p(x_i)p(y_j) \left(\log_2 \frac{1}{p(x_i)} + \log_2 \frac{1}{p(y_j)} \right) \\ &= \sum_{i=1}^M \sum_{j=1}^L \left(p(x_i)p(y_j) \log_2 \frac{1}{p(x_i)} + p(x_i)p(y_j) \log_2 \frac{1}{p(y_j)} \right) \end{aligned}$$

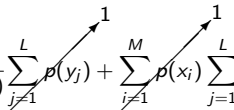
Joint entropy

If X, Y are **independent**

$$\begin{aligned}
 H(X, Y) &= \sum_{i=1}^M \sum_{j=1}^L p(x_i, y_j) \log_2 \frac{1}{p(x_i, y_j)} = \sum_{i=1}^M \sum_{j=1}^L p(x_i)p(y_j) \log_2 \frac{1}{p(x_i)p(y_j)} \\
 &= \sum_{i=1}^M \sum_{j=1}^L p(x_i)p(y_j) \left(\log_2 \frac{1}{p(x_i)} + \log_2 \frac{1}{p(y_j)} \right) \\
 &= \sum_{i=1}^M \sum_{j=1}^L \left(p(x_i)p(y_j) \log_2 \frac{1}{p(x_i)} + p(x_i)p(y_j) \log_2 \frac{1}{p(y_j)} \right) \\
 &= \sum_{i=1}^M \sum_{j=1}^L p(x_i)p(y_j) \log_2 \frac{1}{p(x_i)} + \sum_{i=1}^M \sum_{j=1}^L p(x_i)p(y_j) \log_2 \frac{1}{p(y_j)}
 \end{aligned}$$

Joint entropy

If X, Y are **independent**

$$\begin{aligned}H(X, Y) &= \sum_{i=1}^M \sum_{j=1}^L p(x_i, y_j) \log_2 \frac{1}{p(x_i, y_j)} = \sum_{i=1}^M \sum_{j=1}^L p(x_i)p(y_j) \log_2 \frac{1}{p(x_i)p(y_j)} \\&= \sum_{i=1}^M \sum_{j=1}^L p(x_i)p(y_j) \left(\log_2 \frac{1}{p(x_i)} + \log_2 \frac{1}{p(y_j)} \right) \\&= \sum_{i=1}^M \sum_{j=1}^L \left(p(x_i)p(y_j) \log_2 \frac{1}{p(x_i)} + p(x_i)p(y_j) \log_2 \frac{1}{p(y_j)} \right) \\&= \sum_{i=1}^M \sum_{j=1}^L p(x_i)p(y_j) \log_2 \frac{1}{p(x_i)} + \sum_{i=1}^M \sum_{j=1}^L p(x_i)p(y_j) \log_2 \frac{1}{p(y_j)} \\&= \sum_{i=1}^M p(x_i) \log_2 \frac{1}{p(x_i)} \sum_{j=1}^L p(y_j) + \sum_{i=1}^M p(x_i) \sum_{j=1}^L p(y_j) \log_2 \frac{1}{p(y_j)}\end{aligned}$$


Joint entropy

If X , Y are **independent**

$$\begin{aligned}
 H(X, Y) &= \sum_{i=1}^M \sum_{j=1}^L p(x_i, y_j) \log_2 \frac{1}{p(x_i, y_j)} = \sum_{i=1}^M \sum_{j=1}^L p(x_i) p(y_j) \log_2 \frac{1}{p(x_i) p(y_j)} \\
 &= \sum_{i=1}^M \sum_{j=1}^L p(x_i) p(y_j) \left(\log_2 \frac{1}{p(x_i)} + \log_2 \frac{1}{p(y_j)} \right) \\
 &= \sum_{i=1}^M \sum_{j=1}^L \left(p(x_i) p(y_j) \log_2 \frac{1}{p(x_i)} + p(x_i) p(y_j) \log_2 \frac{1}{p(y_j)} \right) \\
 &= \sum_{i=1}^M \sum_{j=1}^L p(x_i) p(y_j) \log_2 \frac{1}{p(x_i)} + \sum_{i=1}^M \sum_{j=1}^L p(x_i) p(y_j) \log_2 \frac{1}{p(y_j)} \\
 &= \sum_{i=1}^M p(x_i) \log_2 \frac{1}{p(x_i)} \sum_{j=1}^L p(y_j) + \sum_{i=1}^M p(x_i) \sum_{j=1}^L p(y_j) \log_2 \frac{1}{p(y_j)} \\
 &= \underbrace{\sum_{i=1}^M p(x_i) \log_2 \frac{1}{p(x_i)}}_{H(X)} + \underbrace{\sum_{j=1}^L p(y_j) \log_2 \frac{1}{p(y_j)}}_{H(Y)} = H(X) + H(Y)
 \end{aligned}$$

Joint and conditional entropy

$$H(X, Y) = \sum_{i=1}^M \sum_{j=1}^L p(x_i, y_j) \log_2 \frac{1}{p(x_i, y_j)}$$

Joint and conditional entropy

$$H(X, Y) = \sum_{i=1}^M \sum_{j=1}^L p(x_i, y_j) \log_2 \frac{1}{p(x_i, y_j)} = \sum_{i=1}^M \sum_{j=1}^L p(x_i, y_j) \log_2 \frac{1}{p(y_j|x_i)p(x_i)}$$

Joint and conditional entropy

$$\begin{aligned} H(X, Y) &= \sum_{i=1}^M \sum_{j=1}^L p(x_i, y_j) \log_2 \frac{1}{p(x_i, y_j)} = \sum_{i=1}^M \sum_{j=1}^L p(x_i, y_j) \log_2 \frac{1}{p(y_j|x_i)p(x_i)} \\ &= \sum_{i=1}^M \sum_{j=1}^L p(x_i, y_j) \left(\log_2 \frac{1}{p(y_j|x_i)} + \log_2 \frac{1}{p(x_i)} \right) \end{aligned}$$

Joint and conditional entropy

$$\begin{aligned} H(X, Y) &= \sum_{i=1}^M \sum_{j=1}^L p(x_i, y_j) \log_2 \frac{1}{p(x_i, y_j)} = \sum_{i=1}^M \sum_{j=1}^L p(x_i, y_j) \log_2 \frac{1}{p(y_j|x_i)p(x_i)} \\ &= \sum_{i=1}^M \sum_{j=1}^L p(x_i, y_j) \left(\log_2 \frac{1}{p(y_j|x_i)} + \log_2 \frac{1}{p(x_i)} \right) \\ &= \underbrace{\sum_{i=1}^M \sum_{j=1}^L p(x_i, y_j) \log_2 \frac{1}{p(y_j|x_i)}}_{H(Y|X)} + \sum_{i=1}^M \sum_{j=1}^L p(x_i, y_j) \log_2 \frac{1}{p(x_i)} \end{aligned}$$

Joint and conditional entropy

$$\begin{aligned} H(X, Y) &= \sum_{i=1}^M \sum_{j=1}^L p(x_i, y_j) \log_2 \frac{1}{p(x_i, y_j)} = \sum_{i=1}^M \sum_{j=1}^L p(x_i, y_j) \log_2 \frac{1}{p(y_j|x_i)p(x_i)} \\ &= \sum_{i=1}^M \sum_{j=1}^L p(x_i, y_j) \left(\log_2 \frac{1}{p(y_j|x_i)} + \log_2 \frac{1}{p(x_i)} \right) \\ &= \underbrace{\sum_{i=1}^M \sum_{j=1}^L p(x_i, y_j) \log_2 \frac{1}{p(y_j|x_i)}}_{H(Y|X)} + \sum_{i=1}^M \sum_{j=1}^L p(x_i, y_j) \log_2 \frac{1}{p(x_i)} \\ &= H(Y|X) + \sum_{i=1}^M \log_2 \frac{1}{p(x_i)} \underbrace{\sum_{j=1}^L p(x_i, y_j)}_{p(x_i)} = \end{aligned}$$

Joint and conditional entropy

$$\begin{aligned} H(X, Y) &= \sum_{i=1}^M \sum_{j=1}^L p(x_i, y_j) \log_2 \frac{1}{p(x_i, y_j)} = \sum_{i=1}^M \sum_{j=1}^L p(x_i, y_j) \log_2 \frac{1}{p(y_j|x_i)p(x_i)} \\ &= \sum_{i=1}^M \sum_{j=1}^L p(x_i, y_j) \left(\log_2 \frac{1}{p(y_j|x_i)} + \log_2 \frac{1}{p(x_i)} \right) \\ &= \underbrace{\sum_{i=1}^M \sum_{j=1}^L p(x_i, y_j) \log_2 \frac{1}{p(y_j|x_i)}}_{H(Y|X)} + \sum_{i=1}^M \sum_{j=1}^L p(x_i, y_j) \log_2 \frac{1}{p(x_i)} \\ &= H(Y|X) + \sum_{i=1}^M \log_2 \frac{1}{p(x_i)} \underbrace{\sum_{j=1}^L p(x_i, y_j)}_{p(x_i)} = H(Y|X) + \underbrace{\sum_{i=1}^M \log_2 \frac{1}{p(x_i)} p(x_i)}_{H(X)} \end{aligned}$$

Joint and conditional entropy

$$\begin{aligned} H(X, Y) &= \sum_{i=1}^M \sum_{j=1}^L p(x_i, y_j) \log_2 \frac{1}{p(x_i, y_j)} = \sum_{i=1}^M \sum_{j=1}^L p(x_i, y_j) \log_2 \frac{1}{p(y_j|x_i)p(x_i)} \\ &= \sum_{i=1}^M \sum_{j=1}^L p(x_i, y_j) \left(\log_2 \frac{1}{p(y_j|x_i)} + \log_2 \frac{1}{p(x_i)} \right) \\ &= \underbrace{\sum_{i=1}^M \sum_{j=1}^L p(x_i, y_j) \log_2 \frac{1}{p(y_j|x_i)}}_{H(Y|X)} + \sum_{i=1}^M \sum_{j=1}^L p(x_i, y_j) \log_2 \frac{1}{p(x_i)} \\ &= H(Y|X) + \sum_{i=1}^M \log_2 \frac{1}{p(x_i)} \underbrace{\sum_{j=1}^L p(x_i, y_j)}_{p(x_i)} = H(Y|X) + \sum_{i=1}^M \log_2 \frac{1}{p(x_i)} p(x_i) \\ &= H(Y|X) + H(X) \end{aligned}$$

Likewise one can show that

$$H(X, Y) = H(X|Y) + H(Y).$$

Conditional entropies

Since we have both

$$H(X, Y) = H(Y|X) + H(X)$$

and

$$H(X, Y) = H(X|Y) + H(Y)$$

the two right-hand sides above should be equal and we have

$$H(Y|X) = H(X|Y) + H(Y) - H(X)$$

and

$$H(X|Y) = H(Y|X) + H(X) - H(Y).$$

These are not so common/useful.

Mutual information and conditional entropy

$$I(X, Y) = \sum_{i=1}^M \sum_{j=1}^L p(x_i, y_j) \log_2 \frac{p(x_i, y_j)}{p(x_i)p(y_j)}$$

Mutual information and conditional entropy

$$I(X, Y) = \sum_{i=1}^M \sum_{j=1}^L p(x_i, y_j) \log_2 \frac{p(x_i, y_j)}{p(x_i)p(y_j)} = \sum_{i=1}^M \sum_{j=1}^L p(x_i, y_j) \log_2 \frac{p(x_i|y_j)\cancel{p(y_j)}}{p(x_i)\cancel{p(y_j)}}$$

Mutual information and conditional entropy

$$\begin{aligned} I(X, Y) &= \sum_{i=1}^M \sum_{j=1}^L p(x_i, y_j) \log_2 \frac{p(x_i, y_j)}{p(x_i)p(y_j)} = \sum_{i=1}^M \sum_{j=1}^L p(x_i, y_j) \log_2 \frac{p(x_i|y_j)\cancel{p(y_j)}}{p(x_i)\cancel{p(y_j)}} \\ &= \sum_{i=1}^M \sum_{j=1}^L p(x_i, y_j) \left(\log_2 \frac{1}{p(x_i)} + \log_2 p(x_i|y_j) \right) \end{aligned}$$

Mutual information and conditional entropy

$$\begin{aligned} I(X, Y) &= \sum_{i=1}^M \sum_{j=1}^L p(x_i, y_j) \log_2 \frac{p(x_i, y_j)}{p(x_i)p(y_j)} = \sum_{i=1}^M \sum_{j=1}^L p(x_i, y_j) \log_2 \frac{p(x_i|y_j)\cancel{p(y_j)}}{p(x_i)\cancel{p(y_j)}} \\ &= \sum_{i=1}^M \sum_{j=1}^L p(x_i, y_j) \left(\log_2 \frac{1}{p(x_i)} + \log_2 p(x_i|y_j) \right) \\ &= \sum_{i=1}^M \sum_{j=1}^L p(x_i, y_j) \log_2 \frac{1}{p(x_i)} + \sum_{i=1}^M \sum_{j=1}^L p(x_i, y_j) \log_2 p(x_i|y_j) \end{aligned}$$

Mutual information and conditional entropy

$$\begin{aligned} I(X, Y) &= \sum_{i=1}^M \sum_{j=1}^L p(x_i, y_j) \log_2 \frac{p(x_i, y_j)}{p(x_i)p(y_j)} = \sum_{i=1}^M \sum_{j=1}^L p(x_i, y_j) \log_2 \frac{p(x_i|y_j)\cancel{p(y_j)}}{p(x_i)\cancel{p(y_j)}} \\ &= \sum_{i=1}^M \sum_{j=1}^L p(x_i, y_j) \left(\log_2 \frac{1}{p(x_i)} + \log_2 p(x_i|y_j) \right) \\ &= \sum_{i=1}^M \sum_{j=1}^L p(x_i, y_j) \log_2 \frac{1}{p(x_i)} + \sum_{i=1}^M \sum_{j=1}^L p(x_i, y_j) \log_2 p(x_i|y_j) \end{aligned}$$

$$\log_2 x = -\log_2 \frac{1}{x}$$

Mutual information and conditional entropy

$$\begin{aligned}
 I(X, Y) &= \sum_{i=1}^M \sum_{j=1}^L p(x_i, y_j) \log_2 \frac{p(x_i, y_j)}{p(x_i)p(y_j)} = \sum_{i=1}^M \sum_{j=1}^L p(x_i, y_j) \log_2 \frac{p(x_i|y_j)\cancel{p(y_j)}}{p(x_i)\cancel{p(y_j)}} \\
 &= \sum_{i=1}^M \sum_{j=1}^L p(x_i, y_j) \left(\log_2 \frac{1}{p(x_i)} + \log_2 p(x_i|y_j) \right) \\
 &= \sum_{i=1}^M \sum_{j=1}^L p(x_i, y_j) \log_2 \frac{1}{p(x_i)} + \sum_{i=1}^M \sum_{j=1}^L p(x_i, y_j) \log_2 p(x_i|y_j) \\
 \stackrel{\log_2 x = -\log_2 \frac{1}{x}}{=} &= \sum_{i=1}^M \log_2 \frac{1}{p(x_i)} \underbrace{\sum_{j=1}^L p(x_i, y_j)}_{p(x_i)} - \sum_{i=1}^M \sum_{j=1}^L p(x_i, y_j) \log_2 \frac{1}{p(x_i|y_j)}
 \end{aligned}$$

Mutual information and conditional entropy

$$\begin{aligned}
 I(X, Y) &= \sum_{i=1}^M \sum_{j=1}^L p(x_i, y_j) \log_2 \frac{p(x_i, y_j)}{p(x_i)p(y_j)} = \sum_{i=1}^M \sum_{j=1}^L p(x_i, y_j) \log_2 \frac{p(x_i|y_j)\cancel{p(y_j)}}{p(x_i)\cancel{p(y_j)}} \\
 &= \sum_{i=1}^M \sum_{j=1}^L p(x_i, y_j) \left(\log_2 \frac{1}{p(x_i)} + \log_2 p(x_i|y_j) \right) \\
 &= \sum_{i=1}^M \sum_{j=1}^L p(x_i, y_j) \log_2 \frac{1}{p(x_i)} + \sum_{i=1}^M \sum_{j=1}^L p(x_i, y_j) \log_2 p(x_i|y_j) \\
 \log_2 x &= -\log_2 \frac{1}{x} \\
 &= \sum_{i=1}^M \log_2 \frac{1}{p(x_i)} \underbrace{\sum_{j=1}^L p(x_i, y_j)}_{p(x_i)} - \sum_{i=1}^M \sum_{j=1}^L p(x_i, y_j) \log_2 \frac{1}{p(x_i|y_j)} \\
 &= \underbrace{\sum_{i=1}^M p(x_i) \log_2 \frac{1}{p(x_i)}}_{H(X)} - \underbrace{\sum_{i=1}^M \sum_{j=1}^L p(x_i, y_j) \log_2 \frac{1}{p(x_i|y_j)}}_{H(X|Y)}
 \end{aligned}$$

Mutual information and conditional entropy

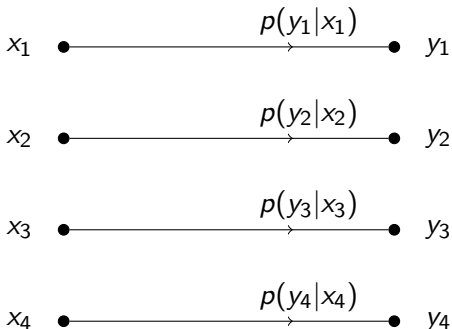
$$\begin{aligned}
 I(X, Y) &= \sum_{i=1}^M \sum_{j=1}^L p(x_i, y_j) \log_2 \frac{p(x_i, y_j)}{p(x_i)p(y_j)} = \sum_{i=1}^M \sum_{j=1}^L p(x_i, y_j) \log_2 \frac{p(x_i|y_j)\cancel{p(y_j)}}{p(x_i)\cancel{p(y_j)}} \\
 &= \sum_{i=1}^M \sum_{j=1}^L p(x_i, y_j) \left(\log_2 \frac{1}{p(x_i)} + \log_2 p(x_i|y_j) \right) \\
 &= \sum_{i=1}^M \sum_{j=1}^L p(x_i, y_j) \log_2 \frac{1}{p(x_i)} + \sum_{i=1}^M \sum_{j=1}^L p(x_i, y_j) \log_2 p(x_i|y_j) \\
 &\stackrel{\log_2 x = -\log_2 \frac{1}{x}}{=} \sum_{i=1}^M \log_2 \frac{1}{p(x_i)} \underbrace{\sum_{j=1}^L p(x_i, y_j)}_{p(x_i)} - \sum_{i=1}^M \sum_{j=1}^L p(x_i, y_j) \log_2 \frac{1}{p(x_i|y_j)} \\
 &= \underbrace{\sum_{i=1}^M p(x_i) \log_2 \frac{1}{p(x_i)}}_{H(X)} - \underbrace{\sum_{i=1}^M \sum_{j=1}^L p(x_i, y_j) \log_2 \frac{1}{p(x_i|y_j)}}_{H(X|Y)} = H(X) - H(X|Y)
 \end{aligned}$$

Likewise, it can be shown that

$$I(X, Y) = H(Y) - H(Y|X).$$

Mutual information: example

Let us consider the DMC



$p(x_1), p(x_2), p(x_3), p(x_4)$ are known

$X \in \{x_1, x_2, x_3, x_4\}$ is transmitted, $Y \in \{y_1, y_2, y_3, y_4\}$ is received
and, in this case, there is one-to-one mapping between X and Y .

Here the conditional probabilities are

$$\begin{aligned}p(y_i|x_i) &= 1 \quad \forall i, \\p(y_j|x_i) &= 0 \quad \forall i \neq j.\end{aligned}$$

The entropy of X is given by

$$H(X) = \sum_{i=1}^4 p(x_i) \log_2 \frac{1}{p(x_i)}.$$

Since $p(y_i) = p(x_i)$ we have

$$H(Y) = \sum_{i=1}^4 p(y_i) \log_2 \frac{1}{p(y_i)} = \sum_{i=1}^4 p(x_i) \log_2 \frac{1}{p(x_i)} = H(X)$$

$$I(X, Y) = \sum_{i=1}^4 \sum_{j=1}^4 p(x_i, y_j) \log_2 \frac{p(x_i, y_j)}{p(x_i)p(y_j)}$$

$$\begin{aligned} I(X, Y) &= \sum_{i=1}^4 \sum_{j=1}^4 p(x_i, y_j) \log_2 \frac{p(x_i, y_j)}{p(x_i)p(y_j)} \\ &= \sum_{i=1}^4 \sum_{j=1}^4 p(y_j|x_i)p(x_i) \log_2 \frac{p(y_j|x_i)\cancel{p(x_i)}}{\cancel{p(x_i)}p(y_j)} \end{aligned}$$

$$\begin{aligned} I(X, Y) &= \sum_{i=1}^4 \sum_{j=1}^4 p(x_i, y_j) \log_2 \frac{p(x_i, y_j)}{p(x_i)p(y_j)} \\ &= \sum_{i=1}^4 \sum_{j=1}^4 p(y_j|x_i)p(x_i) \log_2 \frac{p(y_j|x_i)\cancel{p(x_i)}}{\cancel{p(x_i)}p(y_j)} \\ &\quad \underbrace{p(y_j|x_i)=0}_{\forall j \neq i} \end{aligned}$$

$$\begin{aligned} I(X, Y) &= \sum_{i=1}^4 \sum_{j=1}^4 p(x_i, y_j) \log_2 \frac{p(x_i, y_j)}{p(x_i)p(y_j)} \\ &= \sum_{i=1}^4 \sum_{j=1}^4 p(y_j|x_i)p(x_i) \log_2 \frac{p(y_j|x_i)\cancel{p(x_i)}}{\cancel{p(x_i)}p(y_j)} \\ &\stackrel{\substack{p(y_j|x_i)=0 \\ \forall j \neq i}}{=}}{\quad} \sum_{i=1}^4 p(y_i|x_i)p(x_i) \log_2 \frac{p(y_i|x_i)}{p(y_i)} \end{aligned}$$

$$\begin{aligned} I(X, Y) &= \sum_{i=1}^4 \sum_{j=1}^4 p(x_i, y_j) \log_2 \frac{p(x_i, y_j)}{p(x_i)p(y_j)} \\ &= \sum_{i=1}^4 \sum_{j=1}^4 p(y_j|x_i)p(x_i) \log_2 \frac{p(y_j|x_i)\cancel{p(x_i)}}{\cancel{p(x_i)}p(y_j)} \\ &\stackrel{\substack{p(y_j|x_i)=0 \\ \forall j \neq i}}{=}}{\quad} \sum_{i=1}^4 p(y_i|x_i)p(x_i) \log_2 \frac{p(y_i|x_i)}{p(y_i)} \\ &\stackrel{\substack{p(y_i|x_i)=1 \\ \underline{\quad}}}{=}}{\quad} \end{aligned}$$

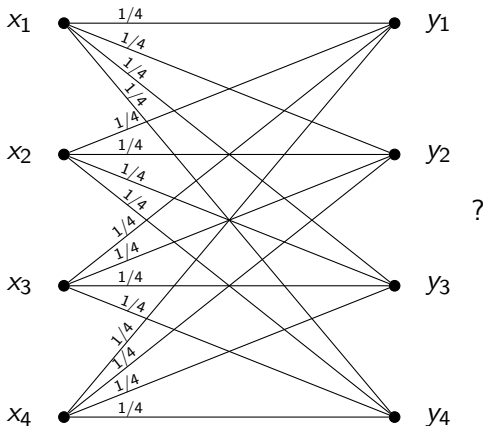
$$\begin{aligned} I(X, Y) &= \sum_{i=1}^4 \sum_{j=1}^4 p(x_i, y_j) \log_2 \frac{p(x_i, y_j)}{p(x_i)p(y_j)} \\ &= \sum_{i=1}^4 \sum_{j=1}^4 p(y_j|x_i)p(x_i) \log_2 \frac{p(y_j|x_i)\cancel{p(x_i)}}{\cancel{p(x_i)}p(y_j)} \\ &\stackrel{\substack{p(y_j|x_i)=0 \\ \forall j \neq i}}{=}}{\sum_{i=1}^4} p(y_i|x_i)p(x_i) \log_2 \frac{p(y_i|x_i)}{p(y_i)} \\ &\stackrel{p(y_i|x_i)=1}{=} \sum_{i=1}^4 p(x_i) \log_2 \frac{1}{p(y_i)} \end{aligned}$$

$$\begin{aligned} I(X, Y) &= \sum_{i=1}^4 \sum_{j=1}^4 p(x_i, y_j) \log_2 \frac{p(x_i, y_j)}{p(x_i)p(y_j)} \\ &= \sum_{i=1}^4 \sum_{j=1}^4 p(y_j|x_i)p(x_i) \log_2 \frac{p(y_j|x_i)\cancel{p(x_i)}}{\cancel{p(x_i)}p(y_j)} \\ &\stackrel{\substack{p(y_j|x_i)=0 \\ \forall j \neq i}}{=}}{\quad} \sum_{i=1}^4 p(y_i|x_i)p(x_i) \log_2 \frac{p(y_i|x_i)}{p(y_i)} \\ &\stackrel{p(y_i|x_i)=1}{=} \sum_{i=1}^4 p(x_i) \log_2 \frac{1}{p(y_i)} \\ &\stackrel{p(y_i)=p(x_i)}{=} \end{aligned}$$

$$\begin{aligned}
 I(X, Y) &= \sum_{i=1}^4 \sum_{j=1}^4 p(x_i, y_j) \log_2 \frac{p(x_i, y_j)}{p(x_i)p(y_j)} \\
 &= \sum_{i=1}^4 \sum_{j=1}^4 p(y_j|x_i)p(x_i) \log_2 \frac{p(y_j|x_i)\cancel{p(x_i)}}{\cancel{p(x_i)}p(y_j)} \\
 &\stackrel{\substack{p(y_j|x_i)=0 \\ \forall j \neq i}}{=}}{\sum_{i=1}^4} p(y_i|x_i)p(x_i) \log_2 \frac{p(y_i|x_i)}{p(y_i)} \\
 &\stackrel{p(y_i|x_i)=1}{=} \sum_{i=1}^4 p(x_i) \log_2 \frac{1}{p(y_i)} \\
 &\stackrel{p(y_i)=p(x_i)}{=} \sum_{i=1}^4 p(x_i) \log_2 \frac{1}{p(x_i)} = H(X) = H(Y)
 \end{aligned}$$

Information at the output of the channel, $I(X, Y)$, is equal to that at the input $H(X)$

What happens if...



Notice

$$p(y_j|x_i) = \frac{1}{4} \forall i, j.$$

$$I(X, Y) = \sum_{i=1}^4 \sum_{j=1}^4 p(x_i, y_j) \log_2 \frac{p(x_i, y_j)}{p(x_i)p(y_j)}$$

$$\begin{aligned} I(X, Y) &= \sum_{i=1}^4 \sum_{j=1}^4 p(x_i, y_j) \log_2 \frac{p(x_i, y_j)}{p(x_i)p(y_j)} \\ &= \sum_{i=1}^4 \sum_{j=1}^4 p(y_j|x_i)p(x_i) \log_2 \frac{p(y_j|x_i)\cancel{p(x_i)}}{\cancel{p(x_i)}p(y_j)} \end{aligned}$$

$$\begin{aligned} I(X, Y) &= \sum_{i=1}^4 \sum_{j=1}^4 p(x_i, y_j) \log_2 \frac{p(x_i, y_j)}{p(x_i)p(y_j)} \\ &= \sum_{i=1}^4 \sum_{j=1}^4 p(y_j|x_i)p(x_i) \log_2 \frac{p(y_j|x_i)\cancel{p(x_i)}}{\cancel{p(x_i)}p(y_j)} \\ &= \sum_{i=1}^4 \sum_{j=1}^4 \frac{1}{4} p(x_i) \log_2 \frac{1/4}{p(y_j)}, \end{aligned} \tag{1}$$

and we need

$$p(y_j) =$$

$$\begin{aligned} I(X, Y) &= \sum_{i=1}^4 \sum_{j=1}^4 p(x_i, y_j) \log_2 \frac{p(x_i, y_j)}{p(x_i)p(y_j)} \\ &= \sum_{i=1}^4 \sum_{j=1}^4 p(y_j|x_i)p(x_i) \log_2 \frac{p(y_j|x_i)\cancel{p(x_i)}}{\cancel{p(x_i)}p(y_j)} \\ &= \sum_{i=1}^4 \sum_{j=1}^4 \frac{1}{4} p(x_i) \log_2 \frac{1/4}{p(y_j)}, \end{aligned} \tag{1}$$

and we need

$$p(y_j) = \sum_{i=1}^4 p(y_j, x_i) =$$

$$\begin{aligned} I(X, Y) &= \sum_{i=1}^4 \sum_{j=1}^4 p(x_i, y_j) \log_2 \frac{p(x_i, y_j)}{p(x_i)p(y_j)} \\ &= \sum_{i=1}^4 \sum_{j=1}^4 p(y_j|x_i)p(x_i) \log_2 \frac{p(y_j|x_i)\cancel{p(x_i)}}{\cancel{p(x_i)}p(y_j)} \\ &= \sum_{i=1}^4 \sum_{j=1}^4 \frac{1}{4} p(x_i) \log_2 \frac{1/4}{p(y_j)}, \end{aligned} \tag{1}$$

and we need

$$p(y_j) = \sum_{i=1}^4 p(y_j, x_i) = \sum_{i=1}^4 p(y_j|x_i)p(x_i) =$$

$$\begin{aligned} I(X, Y) &= \sum_{i=1}^4 \sum_{j=1}^4 p(x_i, y_j) \log_2 \frac{p(x_i, y_j)}{p(x_i)p(y_j)} \\ &= \sum_{i=1}^4 \sum_{j=1}^4 p(y_j|x_i)p(x_i) \log_2 \frac{p(y_j|x_i)\cancel{p(x_i)}}{\cancel{p(x_i)}p(y_j)} \\ &= \sum_{i=1}^4 \sum_{j=1}^4 \frac{1}{4} p(x_i) \log_2 \frac{1/4}{p(y_j)}, \end{aligned} \quad (1)$$

and we need

$$p(y_j) = \sum_{i=1}^4 p(y_j, x_i) = \sum_{i=1}^4 p(y_j|x_i)p(x_i) = \frac{1}{4} \sum_{i=1}^4 p(x_i) = \frac{1}{4},$$

which does not depend on the input!!

Going back to (1),

$$I(X, Y) = \sum_{i=1}^4 \sum_{j=1}^4 \frac{1}{4} p(x_i) \log_2 \frac{1/4}{p(y_j)} = \sum_{i=1}^4 \sum_{j=1}^4 \frac{1}{4} p(x_i) \log_2 \frac{\cancel{1/4}}{\cancel{1/4}} = 0,$$

i.e. **no** information goes through the channel: **Y** does not provide any information about **X** because whatever **X** is transmitted every **Y** is equally likely.

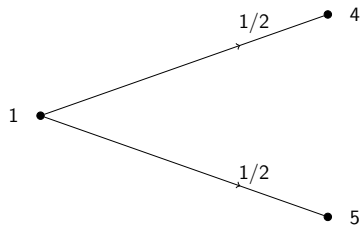
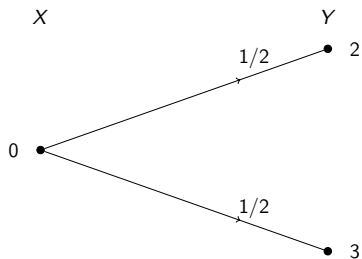
Safety check:

$$p(x_i|y_j) = \frac{p(y_j|x_i)p(x_i)}{p(y_j)} = \frac{1/4 p(x_i)}{1/4} = p(x_i) \text{ (Bayes Theorem).}$$

That is, x_i and y_j are independent

Capacity: example

$$p(X = 0) = \alpha$$
$$p(X = 1) = 1 - \alpha$$



Mathematically,

$$C = \max I(X, Y),$$

Mathematically,

$$C = \max I(X, Y),$$

and, following the usual approach (here, the hard way!!),

$$I(X, Y) = H(Y) - H(Y|X).$$

Mathematically,

$$C = \max I(X, Y),$$

and, following the usual approach (here, the hard way!!),

$$I(X, Y) = H(Y) - H(Y|X).$$

For computing $H(Y)$,

Mathematically,

$$C = \max I(X, Y),$$

and, following the usual approach (here, the hard way!!),

$$I(X, Y) = H(Y) - H(Y|X).$$

For computing $H(Y)$, we need

$$p(Y = 2) = p(2|0)p(X = 0) + \cancel{p(2|1)p(X = 1)} = \frac{1}{2}\alpha$$

$$p(Y = 3) = p(3|0)p(X = 0) + \cancel{p(3|1)p(X = 1)} = \frac{1}{2}\alpha$$

$$p(Y = 4) = \cancel{p(4|0)p(X = 0)} + p(4|1)p(X = 1) = \frac{1}{2}(1 - \alpha)$$

$$p(Y = 5) = \cancel{p(5|0)p(X = 0)} + p(5|1)p(X = 1) = \frac{1}{2}(1 - \alpha).$$

$$\begin{aligned}H(Y) &= p(Y=2) \log_2 \frac{1}{p(Y=2)} + p(3) \log_2 \frac{1}{p(3)} + p(4) \log_2 \frac{1}{p(4)} + p(5) \log_2 \frac{1}{p(5)} \\&= \frac{1}{2} \alpha \log_2 \frac{2}{\alpha} + \frac{1}{2} \alpha \log_2 \frac{2}{\alpha} + \frac{1}{2} (1-\alpha) \log_2 \frac{2}{(1-\alpha)} + \frac{1}{2} (1-\alpha) \log_2 \frac{2}{(1-\alpha)} \\&= \alpha \log_2 \frac{2}{\alpha} + (1-\alpha) \log_2 \frac{2}{(1-\alpha)} \\&= \cancel{\alpha \log_2 2} + \alpha \log_2 \frac{1}{\alpha} + (1-\alpha) \cancel{\log_2 2} + (1-\alpha) \log_2 \frac{1}{1-\alpha} \\&= \cancel{\alpha} + \alpha \log_2 \frac{1}{\alpha} + 1 - \cancel{\alpha} + (1-\alpha) \log_2 \frac{1}{1-\alpha} = 1 + H_b(\alpha).\end{aligned}$$

$$\begin{aligned}
 H(Y) &= p(Y=2) \log_2 \frac{1}{p(Y=2)} + p(3) \log_2 \frac{1}{p(3)} + p(4) \log_2 \frac{1}{p(4)} + p(5) \log_2 \frac{1}{p(5)} \\
 &= \frac{1}{2} \alpha \log_2 \frac{2}{\alpha} + \frac{1}{2} \alpha \log_2 \frac{2}{\alpha} + \frac{1}{2} (1-\alpha) \log_2 \frac{2}{(1-\alpha)} + \frac{1}{2} (1-\alpha) \log_2 \frac{2}{(1-\alpha)} \\
 &= \alpha \log_2 \frac{2}{\alpha} + (1-\alpha) \log_2 \frac{2}{(1-\alpha)} \\
 &= \cancel{\alpha \log_2 2} + \alpha \log_2 \frac{1}{\alpha} + (1-\alpha) \cancel{\log_2 2} + (1-\alpha) \log_2 \frac{1}{1-\alpha} \\
 &= \cancel{\alpha} + \alpha \log_2 \frac{1}{\alpha} + 1 - \cancel{\alpha} + (1-\alpha) \log_2 \frac{1}{1-\alpha} = 1 + H_b(\alpha).
 \end{aligned}$$

$$\begin{aligned}
 H(Y|X) &= \sum_{i=1}^M p(x_i) H(Y|X=x_i) = p(X=0) H(Y|X=0) + p(X=1) H(Y|X=1) \\
 &= \alpha \left(p(2|0) \log_2 \frac{1}{p(2|0)} + p(3|0) \log_2 \frac{1}{p(3|0)} + \cancel{p(4|0) \log_2 \frac{1}{p(4|0)}} + \right. \\
 &\quad \left. + \cancel{p(5|0) \log_2 \frac{1}{p(5|0)}} \right) + (1-\alpha) \left(\cancel{p(2|1) \log_2 \frac{1}{p(2|1)}} + \right. \\
 &\quad \left. + \cancel{p(3|1) \log_2 \frac{1}{p(3|1)}} + p(4|1) \log_2 \frac{1}{p(4|1)} + p(5|1) \log_2 \frac{1}{p(5|1)} \right) \\
 &= \cancel{\alpha \left(\frac{1}{2} \log_2 2 + \frac{1}{2} \log_2 2 \right)} + (1-\alpha) \left(\cancel{\frac{1}{2} \log_2 2} + \frac{1}{2} \log_2 2 \right) = 1.
 \end{aligned}$$

Putting it all together

$$I(X, Y) = H(Y) - H(Y|X) = 1 + H_b(\alpha) - 1 = H_b(\alpha)$$

Putting it all together

$$I(X, Y) = H(Y) - H(Y|X) = 1 + H_b(\alpha) - 1 = H_b(\alpha)$$

An easier way to obtain $I(X, Y)$:

Putting it all together

$$I(X, Y) = H(Y) - H(Y|X) = 1 + H_b(\alpha) - 1 = H_b(\alpha)$$

An easier way to obtain $I(X, Y)$:

$$I(X, Y) = H(X) - H(X|Y) = H(X) = H_b(\alpha),$$

where it has been used that

Putting it all together

$$I(X, Y) = H(Y) - H(Y|X) = 1 + H_b(\alpha) - 1 = H_b(\alpha)$$

An easier way to obtain $I(X, Y)$:

$$I(X, Y) = H(X) - H(X|Y) = H(X) = H_b(\alpha),$$

where it has been used that

- $H(X|Y) = 0$ because it is *the uncertainty we have about X once we know Y*

Putting it all together

$$I(X, Y) = H(Y) - H(Y|X) = 1 + H_b(\alpha) - 1 = H_b(\alpha)$$

An easier way to obtain $I(X, Y)$:

$$I(X, Y) = H(X) - H(X|Y) = H(X) = H_b(\alpha),$$

where it has been used that

- $H(X|Y) = 0$ because it is *the uncertainty we have about X once we know Y*
- $H(X) = H_b(\alpha)$ because X is a binary r.v.

Putting it all together

$$I(X, Y) = H(Y) - H(Y|X) = 1 + H_b(\alpha) - 1 = H_b(\alpha)$$

An easier way to obtain $I(X, Y)$:

$$I(X, Y) = H(X) - H(X|Y) = H(X) = H_b(\alpha),$$

where it has been used that

- $H(X|Y) = 0$ because it is *the uncertainty we have about X once we know Y*
- $H(X) = H_b(\alpha)$ because X is a binary r.v.

The **capacity** is then

$$C = \max I(X, Y) = \max H_b(\alpha) = 1 \frac{\text{bit}}{\text{channel use}},$$

and it is attained when $\alpha = 1/2$ (uniform distribution).