Analog modulations

Communication Theory

Manuel A. Vázquez

May 6, 2024

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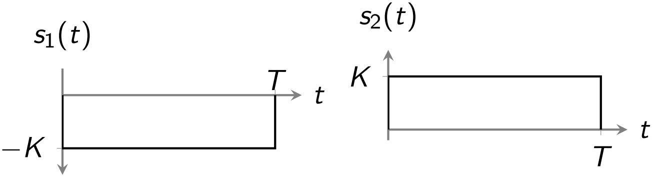
Analog vs digital communications systems

In digital systems, information is carried by symbols in an alphabet,



Alphabet example

Analog vs digital communications systems

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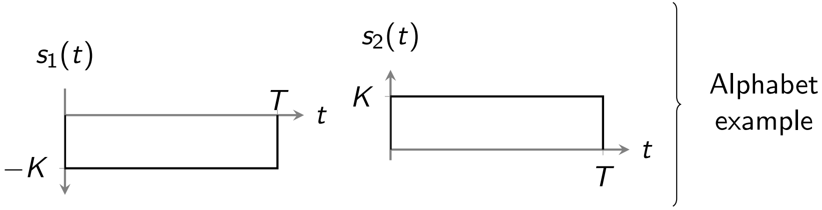
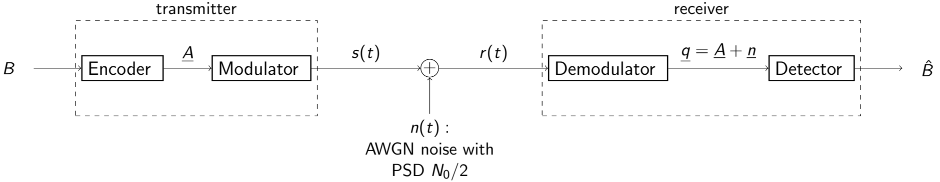
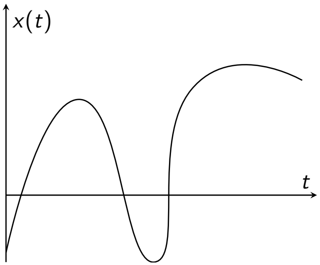


and the most basic scheme of a system is...



Analog vs digital communications systems

In analog systems, information is contained in a continuous waveform...



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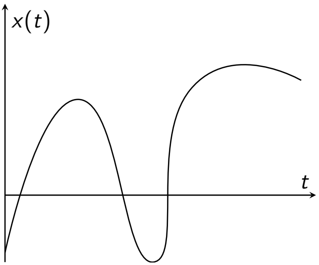

2 possibilities:

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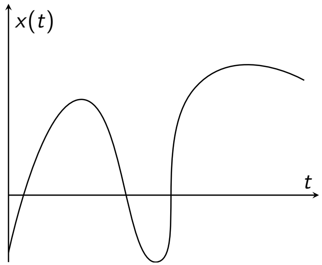

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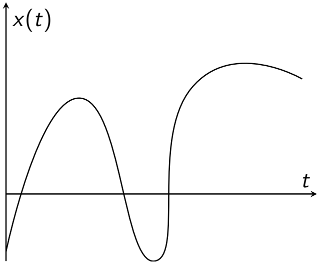

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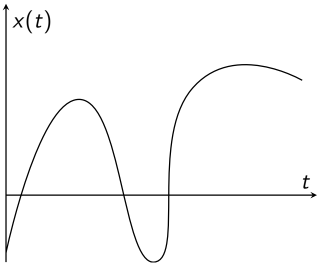

2 possibilities:

- discretize the signal and transmit it using a digital system
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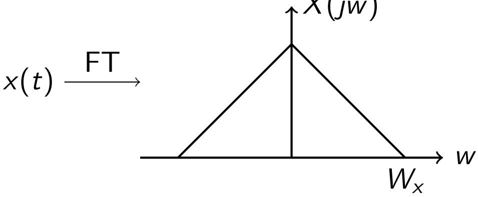
We focus on the latter approach...2 types of channels:

- Baseband
- Passband

Transmission of a baseband signal

Assume we want to transmit

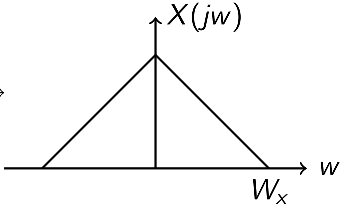


- baseband signal with
- bandwidth W_x

Transmission of a baseband signal

Assume we want to transmit

$x(t) \xrightarrow{\text{FT}}$



- baseband signal with
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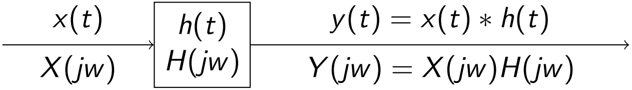
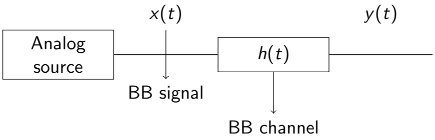
...and we model the channel as a **L**inear **T**ime **I**nvariant system...



Baseband transmission

The information signal is transmitted as is



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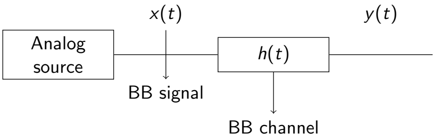
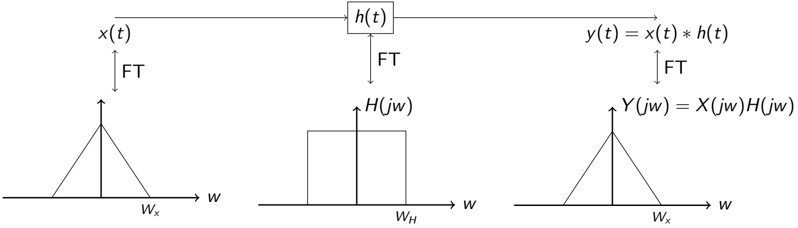


Assuming an ideal baseband channel whose bandwidth is larger than that of the signal ($W_H > W_x$)...



Examples: telephone subscriber loop, public address systems, closed-circuit TV

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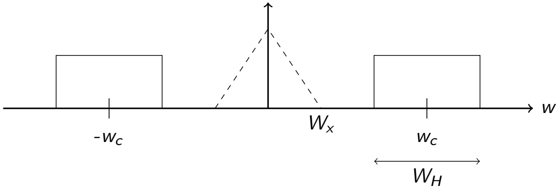

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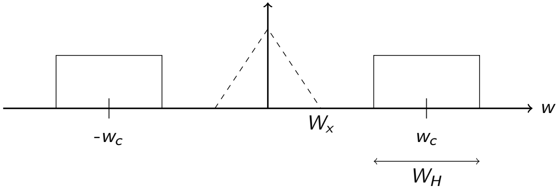

- passband channel with
- bandwidth W_H

2 entails $w_c - \frac{W_H}{2} > W_x$, and hence

$$Y(jw) = X(jw)H(jw) = 0,$$

i.e., the signal **cannot** get through.

Modulation

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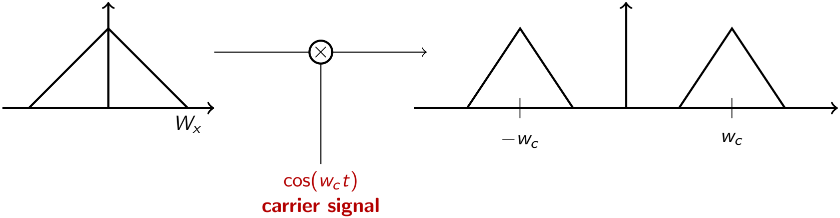

Mathematically,

$$x(t)\cos(w_c t) \xleftrightarrow{FT} \frac{1}{2}X(j(w-w_c)) + \frac{1}{2}X(j(w+w_c))$$

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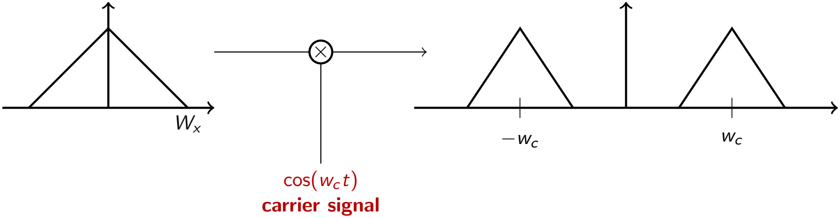

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$$x(t)\cos(w_c t) \xleftrightarrow{FT} \frac{1}{2}X(j(w-w_c)) + \frac{1}{2}X(j(w+w_c))$$

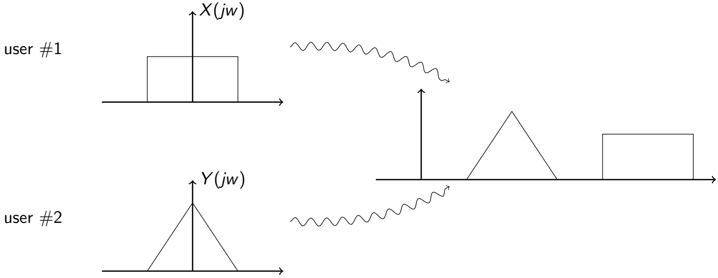
This operation is called **modulation**. It involves three signals

$$\underbrace{y(t)}_{\text{modulated signal}} = \underbrace{x(t)}_{\text{modulating signal}} \cdot \underbrace{\cos(\omega_c t)}_{\text{carrier signal}}$$

Modulation

Other uses:

- multiuser systems



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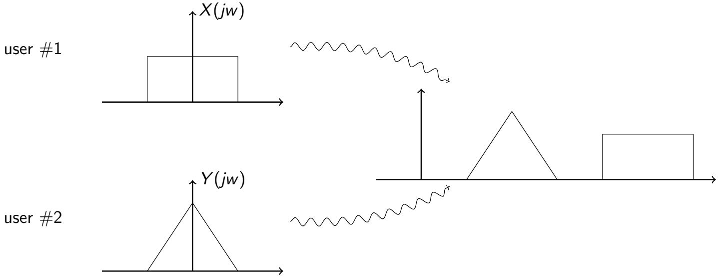

- protection against
  - noise
  - unauthorized users listening in the channel

Types of modulation

Let us consider an information signal, $x(t)$, that is

- a baseband signal, i.e., $X(jw) = 0, \forall\, |w| > W$
  - a realization of a band-limited WSS random process $X(t)$ with $S_X(jw) = 0, \forall\, |w| > W$

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- a *power signal*, i.e., whose power is finite

Transmission of $x(t)$ is achieved by embedding it in a carrier signal of the form

$$\underbrace{A_c}_{\text{amplitude}} \cos(\underbrace{w_c}_{\text{frequency}}\, t + \underbrace{\phi_c}_{\text{phase}})$$

and $x(t)$ can be *modulating*

- the amplitude,
- the frequency, or
- the phase.

Linear vs. angular modulations

In general, the modulated signal has the form

$$y(t) = r(t)\cos(w_c t + \varphi(t))$$

[1] $\nRightarrow r(t) = x(t)$...since some transformation might be applied

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  with *constant* frequency and phase.
- $x(t)$ is embedded in $\varphi(t) \longrightarrow$ **angular modulation**

  $$y(t) = A\cos(w_c t + \varphi(t))$$

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Linear modulations

$$
\begin{aligned}
y(t) &= r(t)\cos(w_c t + \varphi) = r(t)\left(\cos(w_c t)\cos\varphi - \sin(w_c t)\sin\varphi\right) \\
&= \underbrace{r(t)\cos\varphi}_{\substack{x_i(t) \\ \text{in-phase component}}} \cos(w_c t) - \underbrace{r(t)\sin\varphi}_{\substack{x_q(t) \\ \text{quadrature component}}} \sin(w_c t),
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$x_i(t)$ and $x_q(t)$ are rewritten in a more convenient form

- $x_i(t) = r(t)\cos\varphi = A_c + A_m x(t)$
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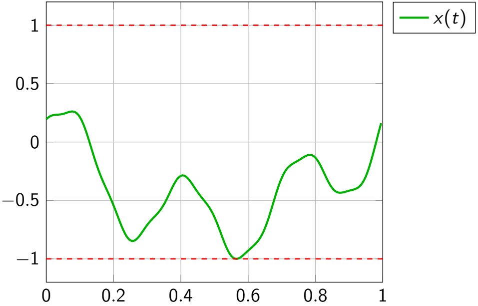
- $A_c = 0,\ A_m \neq 0,\ A_n \neq 0 \rightarrow$ SSB modulation (Single Side Band modulation)

AM modulation

$$y(t) = (A_c + A_m x(t)) \cos(w_c t)$$

Let us assume $|x(t)| \leq 1$ (if not, we can do $x_n(t) = \frac{x(t)}{\max|x(t)|}$,)



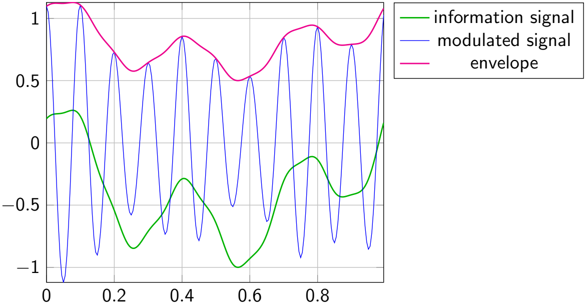
$$y(t) = \left(A_c + \frac{A_c}{A_c} A_m x(t)\right) \cos(w_c t) = A_c (1 + m x(t)) \cos(w_c t)$$

with

$$m = \frac{A_m}{A_c} \equiv \textbf{modulation index}$$

AM demodulation

$A_c\,(1+mx(t)) \geq 0$ implies this signal matches the envelope of the modulated signal, $y(t)$



Envelope of a signal

The **envelope** of an oscillating signal (for example, a cosine) is a *smooth* signal that outlines its extremes.

AM demodulation

In order to guarantee $A_c\,(1 + mx(t)) \geq 0$, and assuming $|x(t)| \leq 1$ (i.e. it is *normalized*), it is enough to choose

$$0 < m \leq 1.$$

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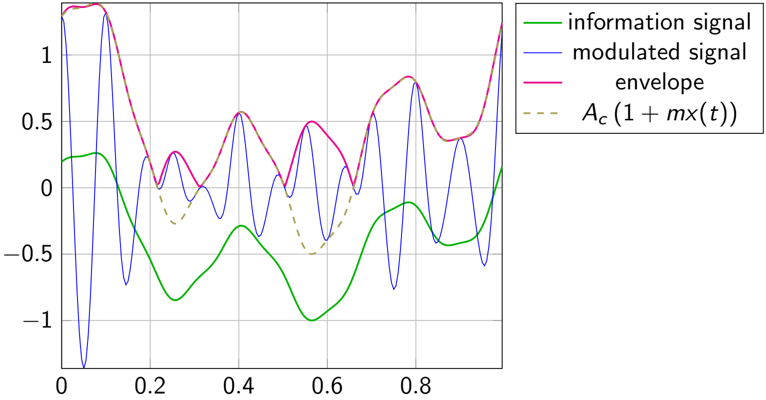
$$y(t) \xrightarrow{\text{extract envelope}} A_c(1 + mx(t)) \xrightarrow{\text{solve for } x(t)} x(t)$$

Overmodulation

If $m > 1$, it might happen that $A_c\,(1 + mx(t)) < 0$ at some point...



The envelope is **not** $A_c\,(1 + mx(t))$ anymore, and hence demodulation is not so easy. It is called **overmodulation**.

Angular modulations

The information is in the argument of the carrier

Drawbacks

- they are way **more complex** than linear modulations
  - sometimes, they need to be studied through approximations
- they take up **more bandwidth**

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Advantages

- they are **less affected by noise**...
  - they trade off bandwidth for immunity against noise

Algebra for Angular modulations

$$y(t) = A\cos\left(w_c t + \varphi(t)\right) = A\cos\phi(t),$$

where

$$\phi(t) = w_c t + \varphi(t) \equiv \textit{instantaneous}\ \text{phase}$$

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From it, we have

$$\frac{d\phi(t)}{dt} = w_i(t)\,\frac{\text{rad}}{\text{second}} \equiv \textit{instantaneous}\ \text{frequency},$$

in radians,

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Doing some algebra

$$f_i(t) = \frac{1}{2\pi}\frac{d\phi(t)}{dt} = \frac{1}{2\pi}\frac{d\left(w_c t + \varphi(t)\right)}{dt} \overset{w_c = 2\pi f_c}{=} f_c + \frac{1}{2\pi}\frac{d\varphi(t)}{dt}$$

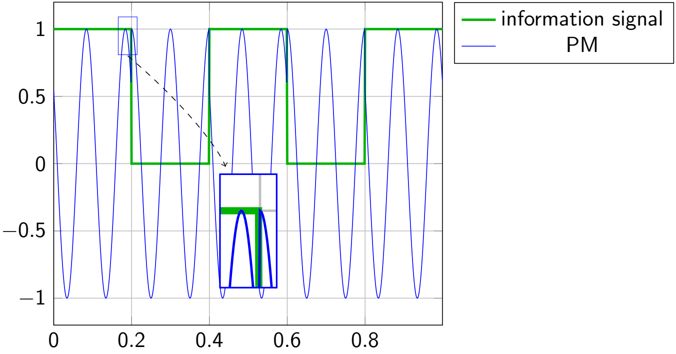
Phase modulation

Being $x(t)$ the information signal...

$$\varphi(t) = \beta x(t) \longrightarrow \text{Phase modulation (PM)}$$

with $\beta \equiv$ phase deviation constant. Hence,

$$y(t) = A\cos(w_c t + \beta x(t))$$



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and hence

$$y(t) = A\cos\left(w_c t + 2\pi f_d \int_{-\infty}^{t} x(u)du\right)$$