Addition of random processes

Manuel A. Vázquez

November 2, 2018

Statement of the problem

$X(t)$ and $Y(t)$ are two processes **jointly stationary**

Reminder

Two processes are **jointly stationary** iff

- they are both (individually) stationary
- their cross-correlation only depends on the time difference.

$$R_{XY}(t_1, t_2) = R_{XY}(\tau)$$
$$R_{YX}(t_1, t_2) = R_{YX}(\tau)$$

We want to know the statistical properties of

$$Z(t) = X(t) + Y(t)$$

Mean and autocorrelation function of $Z(t)$

$$\begin{aligned}\mu_Z(t) &= \mathbb{E}\left[Z(t)\right] = \mathbb{E}\left[X(t) + Y(t)\right] \\ &= \mathbb{E}\left[X(t)\right] + \mathbb{E}\left[Y(t)\right] \overset{\mathrm{X(t),Y(t)WSS}}{=} \boxed{\mu_X + \mu_Y},\end{aligned}$$

i.e., $Z(t)$ is mean-stationary

$$\begin{aligned}R_Z(t+\tau,t) &= \mathbb{E}\left[Z(t+\tau)Z(t)\right] = \mathbb{E}\left[\left(X(t+\tau)+Y(t+\tau)\right)\left(X(t)+Y(t)\right)\right] \\ &= \mathbb{E}\left[X(t+\tau)X(t)\right] + \mathbb{E}\left[X(t+\tau)Y(t)\right] + \mathbb{E}\left[Y(t+\tau)X(t)\right] + \\ &\quad + \mathbb{E}\left[Y(t+\tau)Y(t)\right] \\ &= \boxed{R_X(\tau) + R_{XY}(\tau) + R_{YX}(\tau) + R_Y(\tau)},\end{aligned}$$

i.e., $Z(t)$ is autocorrelation-stationary

Since it is mean- and autocorrelation-stationary... $Z(t)$ is WSS

Power spectral density of $Z(t)$

Since $Z(t)$ is WSS,

$$\begin{aligned} S_Z(j\omega) &= \text{FT}\left[R_Z(\tau)\right] = S_X(j\omega) + S_{XY}(j\omega) + S_{YX}(j\omega) + S_Y(j\omega) \\ &= S_X(j\omega) + \underbrace{S_{XY}(j\omega) + S^*_{XY}(j\omega)}_{\substack{\text{complex number} \\ +\text{ its conjugate}}} + S_Y(j\omega) \\ &= \boxed{S_X(j\omega) + S_Y(j\omega) + 2\text{Re}\left\{S_{XY}(j\omega)\right\}}. \end{aligned}$$

where it has been used that

$$S_{YX}(j\omega) = S^*_{XY}(j\omega)$$

The PSD of $Z(t)$ involves the individual PSD's of $X(t)$ and $Y(t)$...but also the cross-spectral density

The results so far hold as long as $X(t)$ and $Y(t)$ are jointly stationary but...

$X(t)$ and $Y(t)$ uncorrelated

...we can get nicer results if $X(t)$ and $Y(t)$ **uncorrelated**

Reminder

$X(t), Y(t)$ uncorrelated $\iff \text{Cov}\left(X(t+\tau), Y(t)\right) = 0 \, \forall t, \tau.$

The cross-covariance between $X(t)$ and $Y(t)$ is

$$\text{Cov}\left(X(t+\tau), Y(t)\right) = \mathbb{E}\left[\left(X(t+\tau) - \mu_X\right)\left(Y(t) - \mu_Y\right)\right] =$$

$$= \mathbb{E}\left[X(t+\tau)Y(t)\right] - \mu_Y \underbrace{\mathbb{E}\left[X(t+\tau)\right]}_{\mu_X} - \mu_X \underbrace{\mathbb{E}\left[Y(t)\right]}_{\mu_Y} - \mu_X\mu_Y =$$

$$= R_{XY}(\tau) - \mu_X\mu_Y - \cancel{\mu_X\mu_Y} + \cancel{\mu_X\mu_Y} = R_{XY}(\tau) - \mu_X\mu_Y$$

Now,

$X(t), Y(t)$ uncorrelated $\Rightarrow R_{XY}(\tau) - \mu_X\mu_Y = 0 \Rightarrow \boxed{R_{XY}(\tau) = \mu_X\mu_Y}$.

$X(t)$ *or* $Y(t)$ is zero mean

If either $X(t)$ **or** $Y(t)$ is zero mean, then

$$R_{XY}(\tau) = 0$$

which also yields $R_{YX}(\tau) = 0$. In terms of the autocorrelation function, this entails

$$R_Z(\tau) = R_X(\tau) + \cancel{R_{XY}(\tau)} + \cancel{R_{YX}(\tau)} + R_Y(\tau) = R_X(\tau) + R_Y(\tau),$$

and hence,

$$S_Z(j\omega) = \mathrm{FT}\left[R_Z(\tau)\right] = S_X(j\omega) + S_Y(j\omega).$$

Summary

Given

$$Z(t) = X(t) + Y(t),$$

if $X(t)$ and $Y(t)$ satisfy

- jointly stationary
- uncorrelated
- $\mathbb{E}[X(t)] = 0$ **or** $\mathbb{E}[Y(t)] = 0$,

then

$$R_Z(\tau) = R_X(\tau) + R_Y(\tau)$$
$$S_Z(j\omega) = S_X(j\omega) + S_Y(j\omega).$$