



Stochastic processes

Introduction

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January 31, 2024

Stochastic processes

- Extension of the concept of random variable that accounts for temporal dependence

- Random variable

$$\omega \in \Omega \rightarrow X(\omega)$$

- Stochastic process

$$\omega \in \Omega \rightarrow X(t, \omega)$$

- Specifics

- $X(t_i, \omega_j)$: individual realization
- $X(t, \omega_i)$: time-varying signal associated with ω_i , $x_i(t)$
- $X(t_i, \omega)$: random variable ($X(\omega)$)

- Notation: $X(t)$ or $X[n]$

- Interpretation: Indexed set of random variables

- Continuous index ($t \in \mathbb{R}$): Continuous random process
- Discrete index ($n \in \mathbb{Z}$): Discrete random process

Description of a stochastic process

- Analytical description

$$X(t) = f(t, \boldsymbol{\theta})$$

with $\boldsymbol{\theta} = \{\theta_1, \theta_2, \dots, \theta_n\}$ being a vector of random variables.
It entails knowing:

- Mathematical expression for $f(t, \boldsymbol{\theta})$
 - Statistical description of $\boldsymbol{\theta}$
- Statistical description
 - Complete (statistical description): $\forall (t_1, t_2, \dots, t_n) \in \mathbb{R}^n, \forall n$

$$f_{X(t_1), X(t_2), \dots, X(t_n)}(x_1, x_2, \dots, x_n)$$

- M -th order statistics: $\forall n \leq M, \forall (t_1, t_2, \dots, t_n) \in \mathbb{R}^n$

$$f_{X(t_1), X(t_2), \dots, X(t_n)}(x_1, x_2, \dots, x_n)$$

Expectations of the process (statistical averages)

- Mean of a process

$$\mu_X(t) = \mathbb{E}[X(t)] = \int_{-\infty}^{\infty} x f_{X(t)}(x) dx$$

- Autocorrelation function of a process

$$R_X(t_1, t_2) = \mathbb{E}[X(t_1)X(t_2)^*]$$

$$R_X(t_1, t_2) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x_1 x_2 f_{X(t_1), X(t_2)^*}(x_1, x_2) \cdot dx_1 \cdot dx_2$$



Notation

Very often we write $R_X(t_1, t_2)$ as $R_X(t + \tau, t)$ with $t = t_2$ and $\tau = t_1 - t_2$

- Autocovariance of a process

$$\begin{aligned} \text{Cov}_X(t_1, t_2) &= \mathbb{E}[(X(t_1) - \mu_X(t_1))(X(t_2) - \mu_X(t_2))^*] \\ &= R_X(t_1, t_2) - \mu_X(t_1)\mu_X(t_2) \end{aligned}$$

Example of stochastic process I

- Definition of the random experiment
 - Four possible results: $\omega_1, \omega_2, \omega_3, \omega_4$
 - Probabilities: $P(\omega_1) = P(\omega_2) = P(\omega_3) = P(\omega_4) = \frac{1}{4}$
- Analytical description of the process

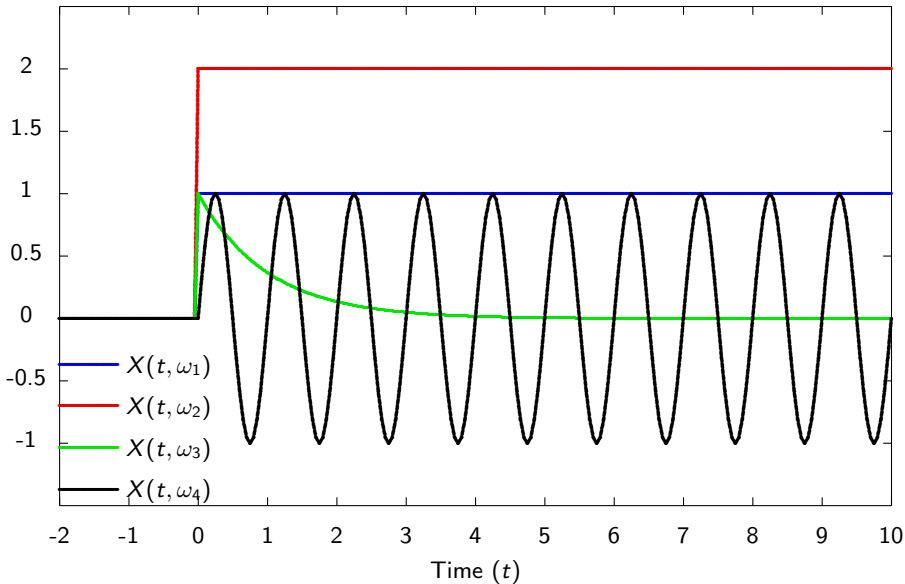
$$X(t, \omega_1) = \begin{cases} 1, & \text{if } t \geq 0 \\ 0, & \text{if } t < 0 \end{cases}$$

$$X(t, \omega_2) = \begin{cases} 2, & \text{if } t \geq 0 \\ 0, & \text{if } t < 0 \end{cases}$$

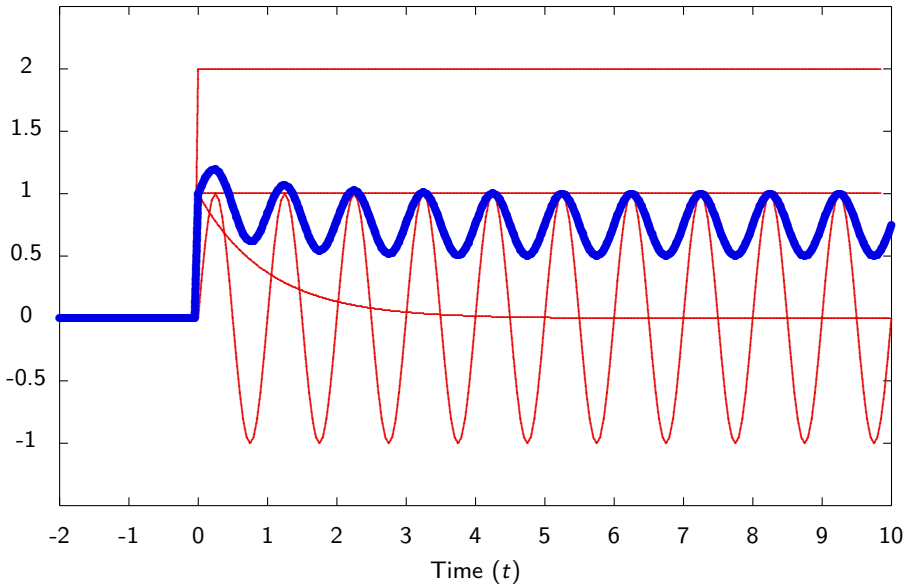
$$X(t, \omega_3) = \begin{cases} e^{-t}, & \text{if } t \geq 0 \\ 0, & \text{if } t < 0 \end{cases}$$

$$X(t, \omega_4) = \begin{cases} \sin(t), & \text{if } t \geq 0 \\ 0, & \text{if } t < 0 \end{cases}$$

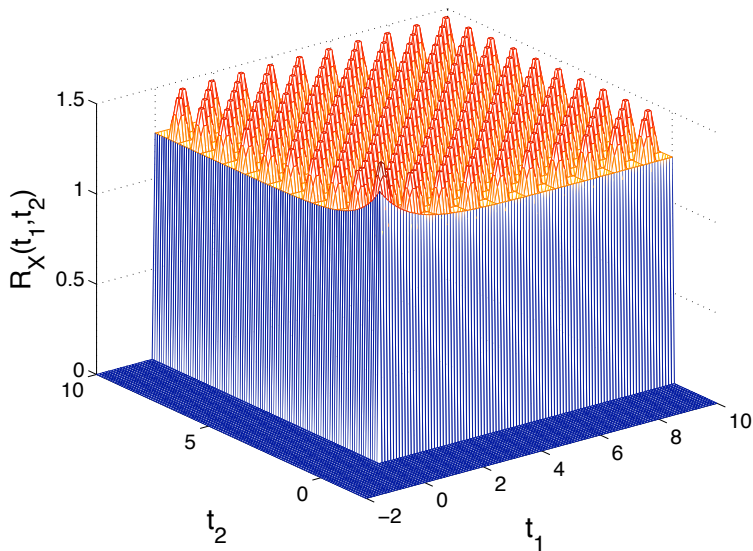
Signals from the random process I



Mean of the random process I



Autocorrelation of the random process I



Example of stochastic process II

- Gaussian random process

$$f_{X(t_1), X(t_2), \dots, X(t_n)}(x_1, x_2, \dots, x_n) = \frac{1}{\sqrt{(2\pi)^n \det(\mathbf{C})}} \times \\ \times e^{-\frac{1}{2}(\mathbf{x}-\boldsymbol{\mu})\mathbf{C}^{-1}(\mathbf{x}-\boldsymbol{\mu})^T}$$

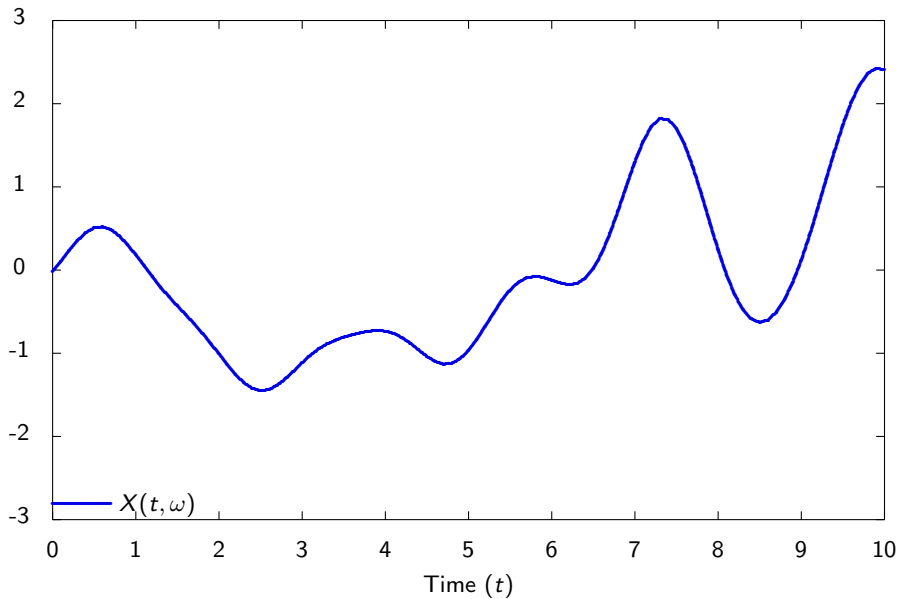
- Vector of means: $\boldsymbol{\mu} = [\mu_1, \mu_2, \dots, \mu_n]^T$

$$\mu_i = E[X(t_i)] = 0$$

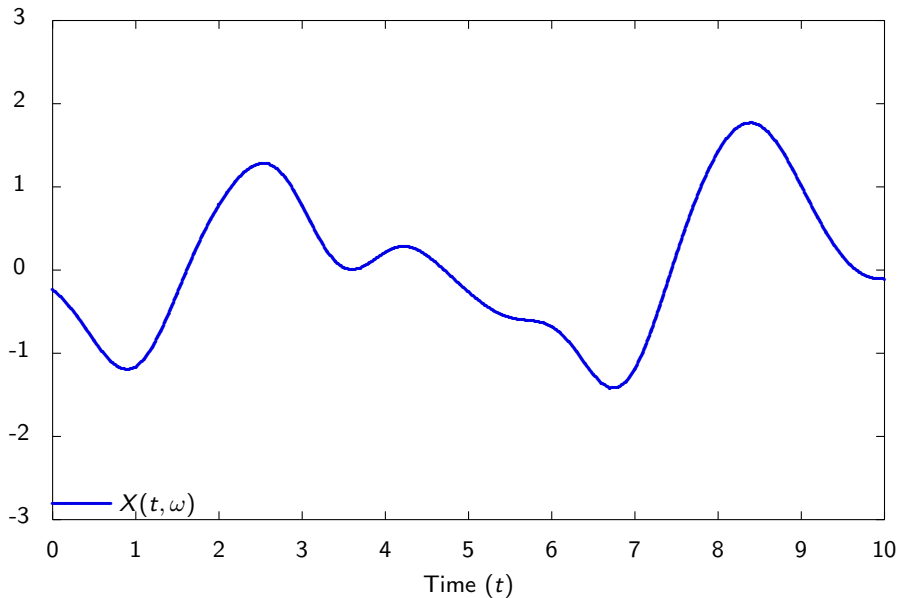
- Matrix of covariances: \mathbf{C} , given by

$$C_{i,j} = \text{Cov}(X(t_i), X(t_j)) = e^{|t_i - t_j|^2}$$

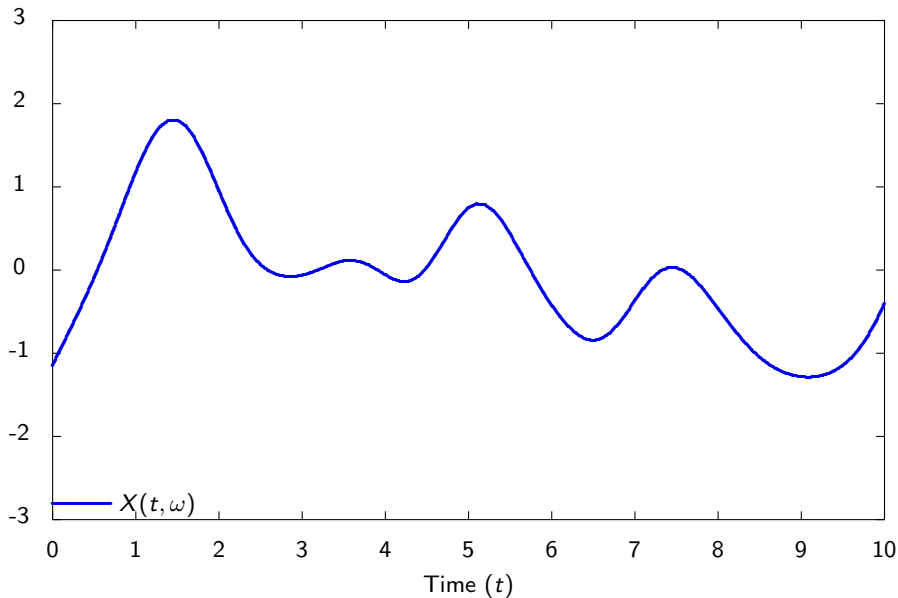
Realization of the random process II



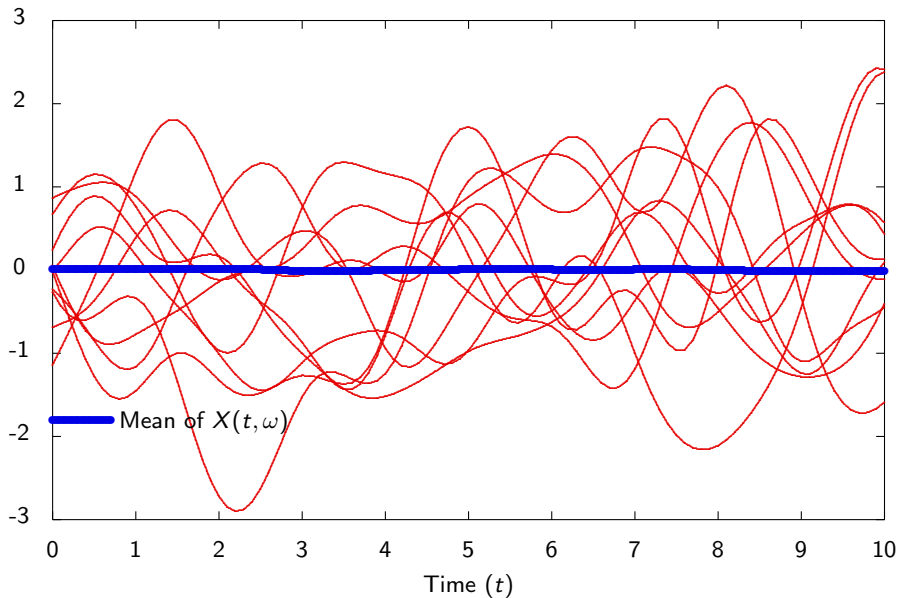
Another realization of the random process II



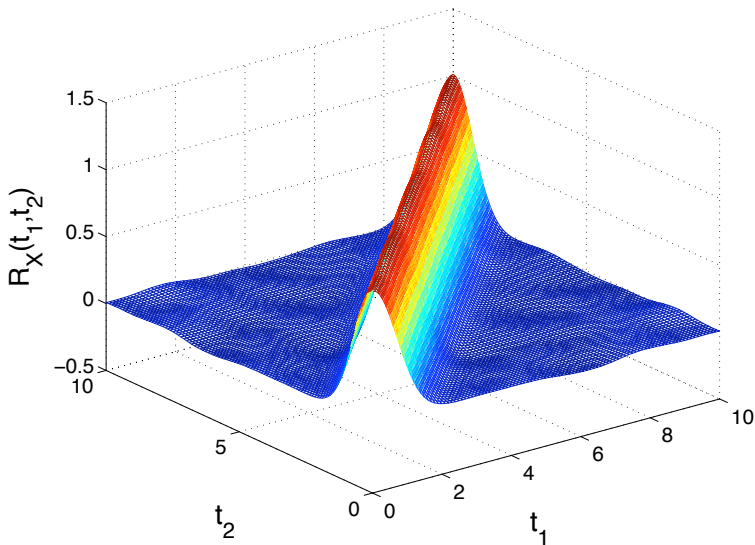
Another realization of the random process II



Mean of the random process II



Autocorrelation of the random process II



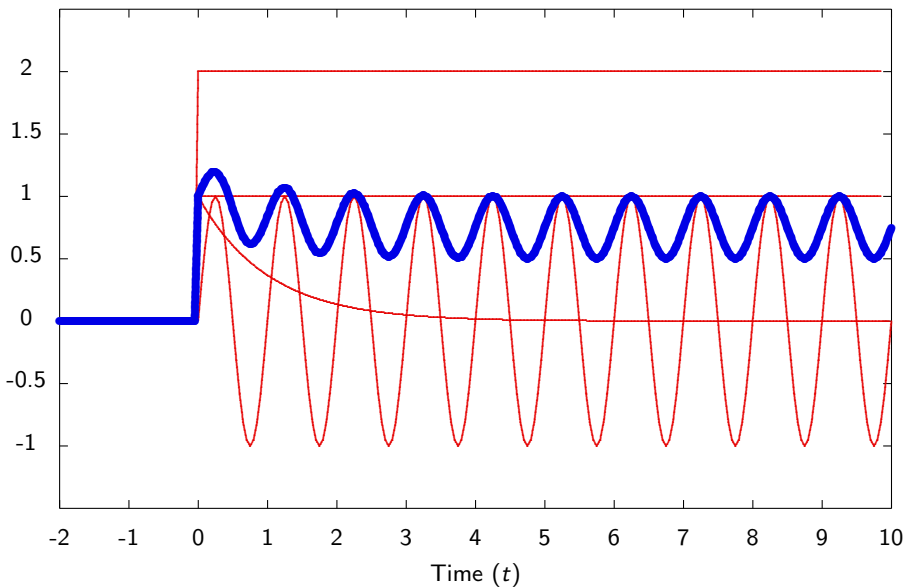
Stationarity

- Strict stationarity: $\forall (t_1, t_2, \dots, t_n), \forall n, \forall \Delta$

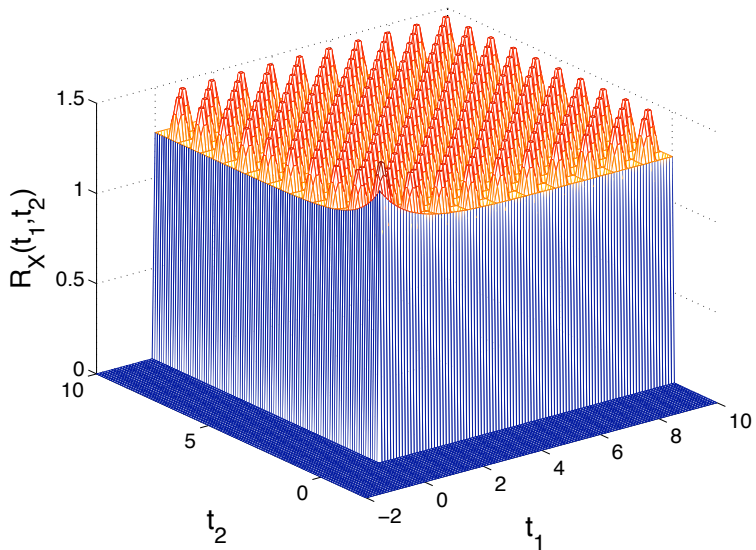
$$f_{X(t_1), X(t_2), \dots, X(t_n)}(x_1, x_2, \dots, x_n) = f_{X(t_1+\Delta), X(t_2+\Delta), \dots, X(t_n+\Delta)}(x_1, x_2, \dots, x_n)$$

- M -th order stationarity: for $n \leq M$
- Wide-sense stationarity
 - 1 $\mu_X(t) = \mu_X$ (it does not depend on t) \Rightarrow **mean-stationary**
 - 2 $R_X(t_1, t_2) = R_X(t_1 - t_2) = R_X(\tau) \Rightarrow$ **autocorrelation-stationary** with $\tau = t_1 - t_2$.

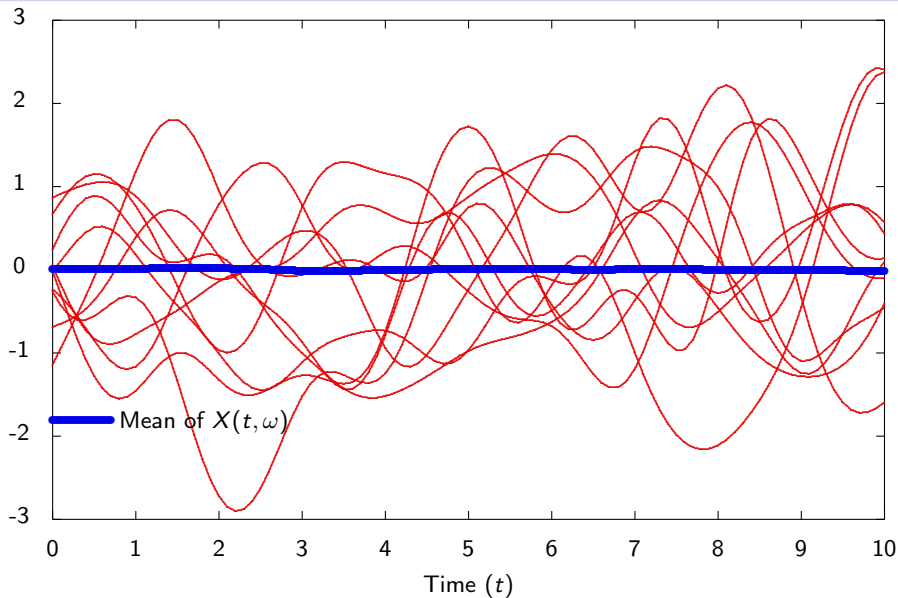
Mean of the random process I



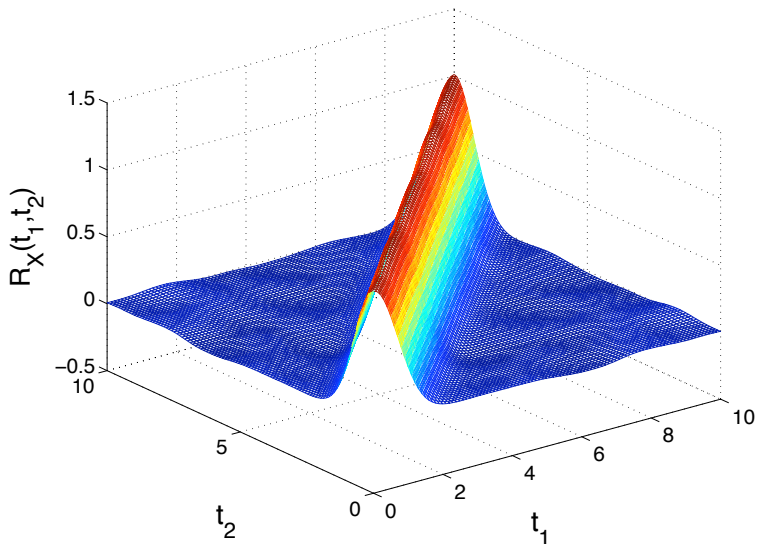
Autocorrelation of the random process I



Mean of the random process II



Autocorrelation of the random process II



Properties of the autocorrelation of WSS processes

The autocorrelation function, $R_X(\tau)$, of a wide-sense stationary process $X(t)$, has the following properties:

- By definition, $\mathbb{E} [X^2(t)] = R_X(0)$
($\mathbb{E} [X^2(t)]$ being the mean square value of the process)
- It is an even function

$$R_X(-\tau) = R_X(\tau)$$

- The modulus is maximum at the origin $\tau = 0$

$$|R_X(\tau)| \leq R_X(0)$$

Expectations of two processes

- Cross-correlation between $X(t)$ and $Y(t)$

$$R_{XY}(t_1, t_2) = \mathbb{E}[X(t_1) \cdot Y(t_2)^*]$$

$$R_{YX}(t_2, t_1) = R_{XY}^*(t_1, t_2)$$

- Cross-covariance between X and Y

$$\text{Cov}_{XY}(t_1, t_2) = \mathbb{E}[(X(t_1) - \mu_X(t_1))(Y(t_2) - \mu_Y(t_2))^*]$$

$$\text{Cov}_{YX}(t_2, t_1) = \text{Cov}_{XY}^*(t_1, t_2)$$

Joint stationarity

Two processes, $X(t)$ and $Y(t)$ are wide-sense jointly stationary if

- they both are individually WSS
- the cross-correlation between them only depends on the time difference, i.e.,

$$R_{XY}(t_1, t_2) = R_{XY}(\tau)$$

with $\tau = t_1 - t_2$