Sensors networks

Non-linear filtering

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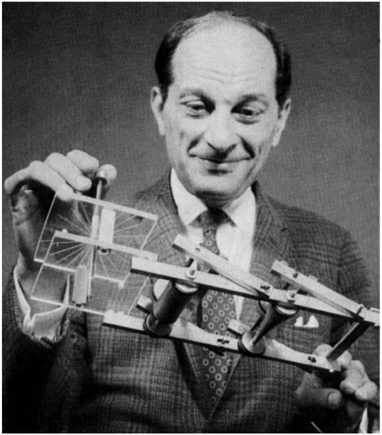
Linearity



> "Using a term like nonlinear science is like referring to the bulk of zoology as the study of non-elephant animals"
>
> — Stanislaw Ulam

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Non-linear dynamic model

We consider the same state equation as before

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...but now the connection between the state and the observations is given by the (vector) function $\mathbf{h} : \mathbb{R}^M \rightarrow \mathbb{R}^N$ (plus additive Gaussian noise like before)

- $$\mathbf{y}_t = \mathbf{h}(\mathbf{x}_t) + \mathbf{w}_t,$$

with $\mathbf{h}$ being a vector of scalar functions of a vector

$$\mathbf{h}(\mathbf{x}_t) = \begin{bmatrix} \mathrm{h}_1(\mathbf{x}_t) \\ \mathrm{h}_2(\mathbf{x}_t) \\ \vdots \\ \mathrm{h}_N(\mathbf{x}_t) \end{bmatrix}$$

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We **cannot** apply the Kalman filter!!

Linearized dynamic model

Goal

To apply the KF over the non-linear model to estimate $\mathbf{x}_t$ given $\mathbf{y}_1, \mathbf{y}_2, \cdots, \mathbf{y}_t$

We can build a linear approximation to the observation equation[1] using a *first-order* Taylor series,

$$\mathbf{h}\left(\mathbf{x}_t\right) \approx \mathbf{h}\left(\mathbf{x}^0\right) + \left[\frac{\partial\, \mathbf{h}}{\partial \mathbf{x}_t}\right]_{\mathbf{x}_t = \mathbf{x}^0} \left(\mathbf{x}_t - \mathbf{x}^0\right),$$

where

$$\left[\frac{\partial\, \mathbf{h}}{\partial \mathbf{x}_t}\right] = \begin{bmatrix} \frac{\partial\, \mathrm{h}_1}{\partial x_{1,t}} & \frac{\partial\, \mathrm{h}_1}{\partial x_{2,t}} & \cdots & \frac{\partial\, \mathrm{h}_1}{\partial x_{M,t}} \\ \frac{\partial\, \mathrm{h}_2}{\partial x_{1,t}} & \frac{\partial\, \mathrm{h}_2}{\partial x_{2,t}} & \cdots & \frac{\partial\, \mathrm{h}_2}{\partial x_{M,t}} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial\, \mathrm{h}_N}{\partial x_{1,t}} & \frac{\partial\, \mathrm{h}_N}{\partial x_{2,t}} & \cdots & \frac{\partial\, \mathrm{h}_N}{\partial x_{M,t}} \end{bmatrix}$$

is the Jacobian matrix (of partial derivatives) of $\mathbf{h}$.

[1] We could do the same thing to deal with a non-linear state equation!!

Deriving the extended Kalman filter

EKF defines the *corrected* observations,

$$\tilde{\mathbf{y}}_t = \mathbf{y}_t - \mathbf{h}\left(\mathbf{x}^0\right) + \left[\frac{\partial\,\mathbf{h}}{\partial \mathbf{x}_t}\right]_{\mathbf{x}_t = \mathbf{x}^0} \mathbf{x}^0,$$

which yield an approximate dynamic model which is both linear and Gaussian

$$\begin{aligned} \mathbf{x}_t &= \mathbf{F}\mathbf{x}_{t-1} + \mathbf{v}_t \\ \tilde{\mathbf{y}}_t &= \left[\frac{\partial\,\mathbf{h}}{\partial \mathbf{x}_t}\right]_{\mathbf{x}_t = \mathbf{x}^0} \mathbf{x}_t + \mathbf{w}_t \end{aligned}$$

Success!!

It is straightforward to apply the KF on the previous model.

Extended Kalman Filter

- Prediction

$$
\begin{aligned}
\hat{\mathbf{x}}_{t|t-1} &= \mathbf{F}\hat{\mathbf{x}}_{t-1|t-1} \\
\mathbf{P}_{t|t-1} &= \mathbf{Q} + \mathbf{F}\mathbf{P}_{t-1|t-1}\mathbf{F}^\top
\end{aligned}
$$

Extended Kalman Filter

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\end{aligned}
$$

- Update

$$
\begin{aligned}
\mathbf{K}_t &= \mathbf{P}_{t|t-1}\left[\frac{\partial \mathbf{h}}{\partial \mathbf{x}_t}\right]^\top_{\mathbf{x}_t=\mathbf{x}^0}\left(\left[\frac{\partial \mathbf{h}}{\partial \mathbf{x}_t}\right]_{\mathbf{x}_t=\mathbf{x}^0}\mathbf{P}_{t|t-1}\left[\frac{\partial \mathbf{h}}{\partial \mathbf{x}_t}\right]^\top_{\mathbf{x}_t=\mathbf{x}^0}+\mathbf{R}\right)^{-1} \\
\hat{\mathbf{x}}_{t|t} &= \hat{\mathbf{x}}_{t|t-1} + \mathbf{K}_t(\mathbf{y}_t - \mathbf{h}(\hat{\mathbf{x}}_{t|t-1})) \\
\mathbf{P}_{t|t} &= \mathbf{P}_{t|t-1} - \mathbf{K}_t\left(\left[\frac{\partial \mathbf{h}}{\partial \mathbf{x}_t}\right]_{\mathbf{x}_t=\mathbf{x}^0}\mathbf{P}_{t|t-1}\left[\frac{\partial \mathbf{h}}{\partial \mathbf{x}_t}\right]^\top_{\mathbf{x}_t=\mathbf{x}^0}\right)\mathbf{K}_n^\top,
\end{aligned}
$$

where $\mathbf{Q}$ is the covariance matrix of $\mathbf{v}_t$, and $\mathbf{R}$ that of $\mathbf{w}_t$.

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\hat{\mathbf{x}}_{t|t} &= \hat{\mathbf{x}}_{t|t-1} + \mathbf{K}_t(\mathbf{y}_t - \mathbf{h}(\hat{\mathbf{x}}_{t|t-1})) \\
\mathbf{P}_{t|t} &= \mathbf{P}_{t|t-1} - \mathbf{K}_t\left(\left[\frac{\partial \mathbf{h}}{\partial \mathbf{x}_t}\right]_{\mathbf{x}_t=\mathbf{x}^0}\mathbf{P}_{t|t-1}\left[\frac{\partial \mathbf{h}}{\partial \mathbf{x}_t}\right]^\top_{\mathbf{x}_t=\mathbf{x}^0}\right)\mathbf{K}_n^\top,
\end{aligned}
$$

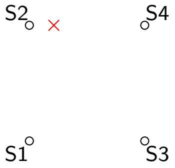
where $\mathbf{Q}$ is the covariance matrix of $\mathbf{v}_t$, and $\mathbf{R}$ that of $\mathbf{w}_t$.

- The linearization point $\mathbf{x}^0$ must be close enough to $\mathbf{x}_t$ for the algorithm to work properly. Usually, we take $\mathbf{x}^0 = \hat{\mathbf{x}}_{t|t-1}$.

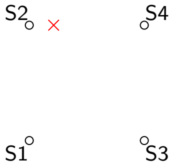
Example of non-linear function for tracking

Goal

Localization and tracking of an object moving with a known constant velocity **c**.



Example of non-linear function for tracking



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Same state equation as before,

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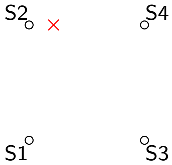


Goal

Localization and tracking of an object moving with a known constant velocity $\mathbf{c}$.

Same state equation as before,

- $$\mathbf{x}_t = \mathbf{x}_{t-1} + \mathbf{c}T + \mathbf{v}_t,$$

...a more *realistic* observation equation based on the Received Signal Strength Indicator (RSSI),

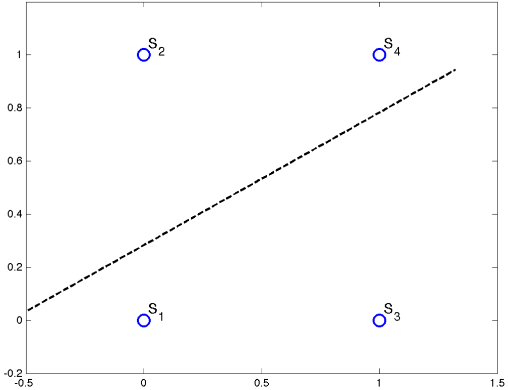
- $$y_{t,i} = \underbrace{k_1 - k_2 \log \|\mathbf{x}_t - \mathbf{s}_i\|}_{\text{RSSI}_i} + w_{t,i}, \quad i = 1, \cdots, N$$

  with k_1 and k_2 being some known constants and $\mathbf{s}_i$ the position of the corresponding sensor.

  (previously, $\mathbf{y}_t = \mathbf{x}_t + \mathbf{w}_t$)

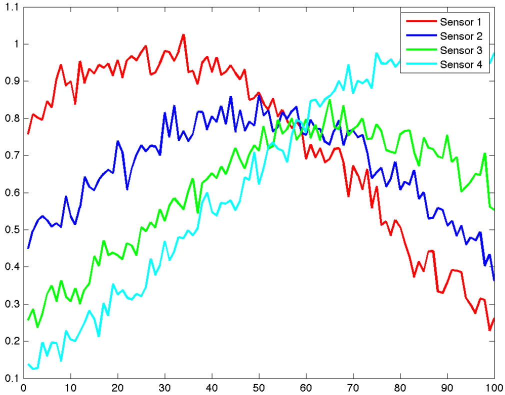
Example

- True trajectory



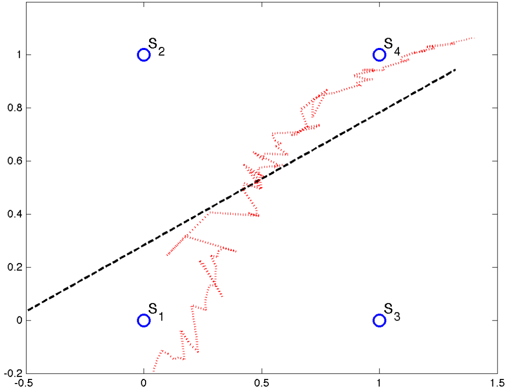
Example

- Sensors readings



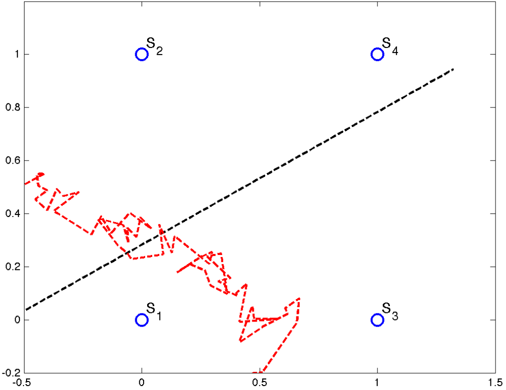
Example

- Result when filtering using only Sensor 2



Example

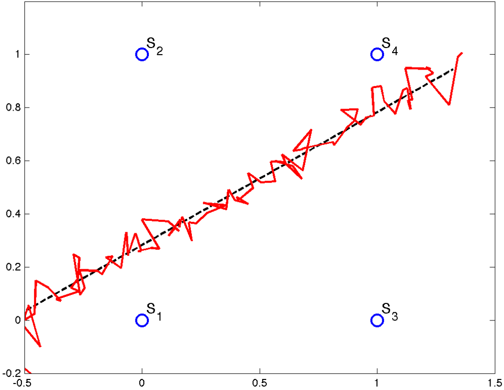
- Result when filtering using Sensors 1 and 2



- Sensors cannot disambiguate the direction.

Example

- Result when filtering using the four sensors



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Unscented Kalman filter

- Another non-linear extension of the Kalman filter (alternative to EKF)...

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- The model:

$$\begin{aligned} \mathbf{x}_t &= \mathbf{f}\left(\mathbf{x}_{t-1}, \mathbf{v}_t\right), \quad \mathbf{v}_t \sim \mathcal{N}\left(\mathbf{0}, \mathbf{Q}_n\right) \\ \mathbf{y}_t &= \mathbf{H}_t \mathbf{x}_t + \mathbf{w}_t, \quad \mathbf{w}_t \sim \mathcal{N}\left(\mathbf{0}, \mathbf{R}_n\right) \end{aligned}$$

The observation equation is linear...but the state equation is **not** ($\mathbf{f}$ is any arbitrary vector function)

Unscented Kalman filter

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  The observation equation is linear...but the state equation is **not** ($\mathbf{f}$ is any arbitrary vector function)
- It relies on the...

unscented transformation

a method for computing the moments of a *Gaussian* random variable that undergoes a nonlinear transformation.

...which in turn makes use of a...

Sigma point representation

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$$
\begin{aligned}
\mathbf{X}_t(0) &= \hat{\mathbf{x}}_t, & \mathrm{W}_t(0) &= \kappa/(M+\kappa) \\
\mathbf{X}_t(i) &= \hat{\mathbf{x}}_t + \left(\sqrt{(M+\kappa)P_t}\right)_i, & \mathrm{W}_t(i) &= 1/\left(2(M+\kappa)\right) \\
\mathbf{X}_t(i+M) &= \hat{\mathbf{x}}_t - \left(\sqrt{(M+\kappa)P_t}\right)_i, & \mathrm{W}_t(i+M) &= 1/\left(2(M+\kappa)\right)
\end{aligned}
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for $i = 1, \cdots, M$, where $\kappa \in \mathbb{R}$ and $\left(\sqrt{(M+\kappa)P_t}\right)_i$ is the i-th column of the matrix square root of $(M+\kappa)\mathbf{P}_t$.

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Theorem: Sigma points

This set of weighted samples has the same sample mean and covariance as the original distribution.

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3. Compute the predictive covariance

$$\mathbf{P}_{t+1}^{-} = \sum_{i=0}^{2M} \mathrm{W}_t(i) \left(\mathbf{X}_{t+1|t}(i) - \hat{\mathbf{x}}_{t+1}^{-}\right) \left(\mathbf{X}_{t+1|t}(i) - \hat{\mathbf{x}}_{t+1}^{-}\right)^{\top}$$

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The *update step* is carried out as in the standard KF.

Remarks

- The mean vector and covariance matrix computed by propagating the sigma points through the nonlinearity are still **estimates**, but more accurate than those produced by the EKF. They are correct up to the 2nd order of a Taylor expansion. ✓

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- Different choices of sigma points are possible. If a Gauss-Hermite quadrature rule is used, a larger number of points is needed but the approximations are more accurate as well.
- UKF algorithms look simple to implement. However performance may actually vary depending, e.g., on the number of points.

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State space models

- Formal statement of the estimation problem...

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- ...in a Bayesian framework.

State space models

- Formal statement of the estimation problem...
- ...in a Bayesian framework.
- Non-linear state space model

$$\left\{ \begin{array}{c} \mathbf{x}_t = \mathbf{f}(\mathbf{x}_{t-1}, \mathbf{v}_t) \\ \mathbf{y}_t = \mathbf{h}(\mathbf{x}_t, \mathbf{w}_t) \end{array} \right\} \Leftrightarrow \left\{ \begin{array}{c} \mathbf{x}_0 \sim p(\mathbf{x}_0) \\ \mathbf{x}_t \sim p(\mathbf{x}_t|\mathbf{x}_{t-1}) \\ \mathbf{y}_t \sim p(\mathbf{y}_t|\mathbf{x}_t) \end{array} \right\}$$

where

- $\mathbf{f}, \mathbf{h} \equiv$ state and observation functions;
- $\mathbf{v}_t, \mathbf{w}_t \equiv$ state and observation noise;
- $p(\mathbf{x}_0) \equiv$ prior pdf of the state;
- $p(\mathbf{x}_t|\mathbf{x}_{t-1}) \equiv$ transition pdf of the state;
- $p(\mathbf{y}_t|\mathbf{x}_t) \equiv$ conditional pdf of the observation (likelihood of the state).

Stochastic filtering

Goal

Tracking the posterior distribution, $p(\mathbf{x}_t|\mathbf{y}_{1:t})$, which allows computing the expectation of any function of interest, $\mathbf{g}$, as

$$\mathbb{E}\left[\mathbf{g}(\mathbf{x}_t)\right] = \int \mathbf{g}(\mathbf{x}_t) p(\mathbf{x}_t|\mathbf{y}_{1:t}) d\mathbf{x}_t$$

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Using Bayes theorem, one can easily show

$$p(\mathbf{x}_t|\mathbf{y}_{1:t}) \propto p(\mathbf{y}_t|\mathbf{x}_t) \int p(\mathbf{x}_t|\mathbf{x}_{t-1}) p(\mathbf{x}_{t-1}|\mathbf{y}_{1:t-1}) d\mathbf{x}_{t-1}$$

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Stochastic filtering

There is uncertainty in the observations and/or the noise governing the evolution of the system...that's why we talk about **stochastic filtering**[a].

[a] Kalman filter also falls within this category!!

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Monte Carlo integration

Let X be a r.v. with pdf $p(x)$ and consider the problem of approximating

$$\mathbb{E}\left[h(X)\right] = \int h(x)p(x)dx$$

for some integrable function h.

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Let X be a r.v. with pdf $p(x)$ and consider the problem of approximating

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One possible approach

If we can **draw** N **i.i.d. samples** $x^{(1)}, ..., x^{(N)}$ from $p(x)$ and the variance of the r.v. $Y = h(X)$ is finite, then

$$\lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} h(X^{(n)}) = \mathbb{E}\left[h(X)\right]$$

almost surely (a.s.).

Sampling

Unfortunately, in many problems it is impossible to draw samples from $p(x)$...

Example

$$\mathbf{y}_t = \mathbf{H}^H \mathbf{x}_t + \mathbf{w}_t$$

We want to estimate $\mathbf{x}_t$ from $\mathbf{y}_t$, i.e., we aim at approximating $p(\mathbf{x}_t \mid \mathbf{y}_t)$...but we cannot sample directly from the latter (how??)

[2] Say $p(x) = Kf(x)$ where function $f(x)$ is known, but constant K is not.

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...but maybe $p(x)$ can be evaluated *up to a proportionality constant*[2]:

$$p(\mathbf{x}_t \mid \mathbf{y}_t) = \frac{p(\mathbf{y}_t \mid \mathbf{x}_t)p(\mathbf{x}_t)}{p(\mathbf{y}_t)} \propto p(\mathbf{y}_t \mid \mathbf{x}_t)p(\mathbf{x}_t)$$

[2] Say $p(x) = Kf(x)$ where function $f(x)$ is known, but constant K is not.

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Importance sampling

Assume the pdf of interest, $p(x)$, (the **target** pdf) can be evaluated up to a proportionality constant and

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Assume the pdf of interest, $p(x)$, (the **target** pdf) can be evaluated up to a proportionality constant and

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$$p(x) > 0 \Rightarrow q(x) > 0$$

Importance sampling

Assume the pdf of interest, $p(x)$, (the **target** pdf) can be evaluated up to a proportionality constant and

- *choose* a pdf, $q(x)$, known as **proposal function** such that

$$p(x) > 0 \Rightarrow q(x) > 0$$

- define the **weight function** as

$$w(x) = c\frac{p(x)}{q(x)}$$

where c is an arbitrary (possibly unknown) constant

Importance sampling

Assume the pdf of interest, $p(x)$, (the **target** pdf) can be evaluated up to a proportionality constant and

- *choose* a pdf, $q(x)$, known as **proposal function** such that

$$p(x) > 0 \Rightarrow q(x) > 0$$

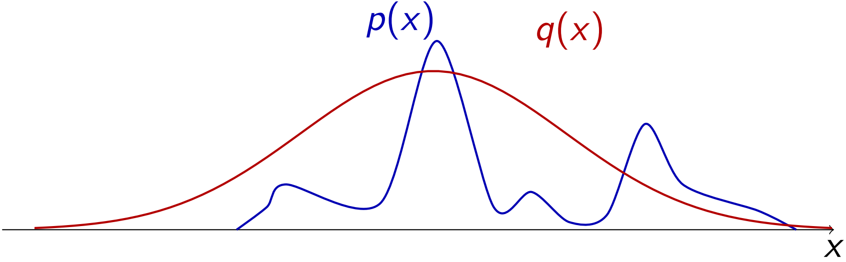
- define the **weight function** as

$$w(x) = c\frac{p(x)}{q(x)}$$

where c is an arbitrary (possibly unknown) constant

then we can compute the expectation of any arbitrary function $h(x)$ with respect to $p(x)$...but using samples from $q(x)$!!

Importance sampling: $q(x)$



Constraint

The support of $q(x)$ must encompass that of $p(x)$,

$$p(x) > 0 \Rightarrow q(x) > 0$$

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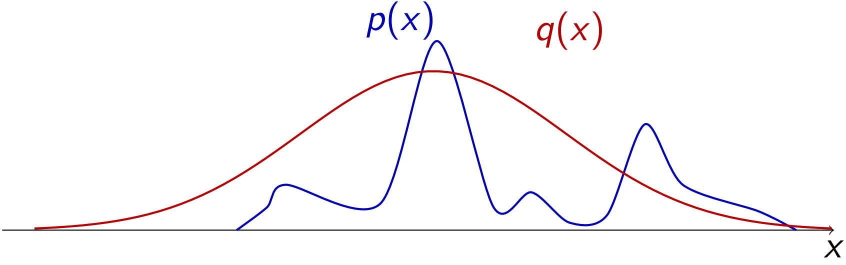

Constraint

The support of $q(x)$ must encompass that of $p(x)$,

$$p(x) > 0 \Rightarrow q(x) > 0$$

How to choose it

For the sake of efficiency, the proposal pdf should be as close as possible to the target pdf.

Importance sampling: procedure

...to approximate $\mathbb{E}[h(X)]$ with respect to $p(x)$ using samples from $q(x)$

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1. Draw $\mathbf{x}^{(i)} \sim q(x)$ for $i = 1, \cdots, N$
2. Compute

$$w(\mathbf{x}^{(i)}) = c\frac{p(\mathbf{x}^{(i)})}{q(\mathbf{x}^{(i)})} \triangleq w^{*(i)} \text{ (\textit{unnormalized} weight)}$$

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$$w^{(i)} = \frac{w^{*(i)}}{\sum_{j=1}^{N} w^{*(j)}}$$

4. Approximate $\mathbb{E}[h(X)]$ as

$$\mathbb{E}[h(X)] \approx \sum_{i=1}^{N} w^{(i)} h(\mathbf{x}^{(i)}) \tag{1}$$

Importance sampling: interpretation

Using IS, we end up with a collection of pairs (`sample`,`weight`):

$$\left\{ \left(\mathbf{x}^{(1)}, w^{(1)}\right), \left(\mathbf{x}^{(2)}, w^{(2)}\right), \left(\mathbf{x}^{(3)}, w^{(3)}\right), \cdots \right\}$$

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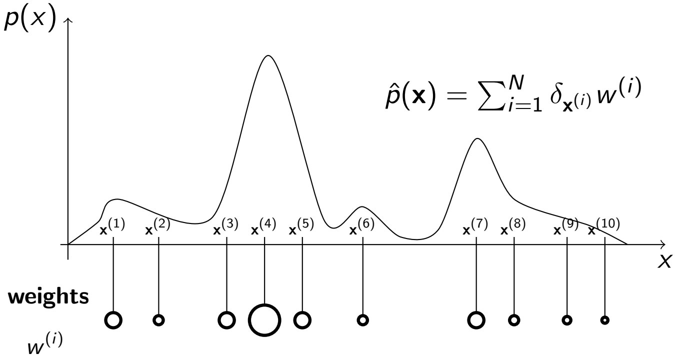
The weight can be *interpreted* as the probability of the corresponding sample



Importance sampling in dynamic systems

? **Recursive IS**

Can we apply IS to **recursively** estimate the state in a dynamic system?

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Let us consider a dynamic model in state-space form specified by

$$\mathbf{x}_0 \sim p(\mathbf{x}_0), \quad \mathbf{x}_t \sim p(\mathbf{x}_t|\mathbf{x}_{t-1}), \quad \mathbf{y}_t \sim p(\mathbf{y}_t|\mathbf{x}_t)$$

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We already know how to approximate **any** distribution of interest, and hence we could approximate

$$p(\mathbf{x}_t \mid \mathbf{y}_{1:t}), p(\mathbf{x}_{t+1} \mid \mathbf{y}_{1:t+1}), p(\mathbf{x}_{t+2} \mid \mathbf{y}_{1:t+2}), \cdots$$

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Goal

Build (using importance sampling) an approximation of $p(\mathbf{x}_{t+1} \mid \mathbf{y}_{1:t+1})$ using one from $p(\mathbf{x}_t \mid \mathbf{y}_{1:t})$.

Index

Importance sampling in dynamic systems

Assume we have an approximation of $p(\mathbf{x}_{t-1} \mid \mathbf{y}_{1:t-1})$ given by

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Particle filtering

- **Initialization**

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  - samples are drawn from the prior,

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  - all the weights are set to the same value

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- **Recursion** at time t
  - draw samples, $\mathbf{x}_t^{(1)}, \mathbf{x}_t^{(2)}, \cdots$, from the *selected* proposal,
$$\mathbf{x}_t^{(i)} \sim q(\mathbf{x}_t \mid \mathbf{y}_{1:t})$$
  - compute the weights
$$w_t^{(i)} \propto \frac{p(\mathbf{x}_t^{(i)} \mid \mathbf{y}_{1:t})}{q(\mathbf{x}_t^{(i)} \mid \mathbf{y}_{1:t})} = \frac{p(\mathbf{y}_t \mid \mathbf{x}_t^{(i)}) \sum_{i=1}^{N} p(\mathbf{x}_t^{(i)} \mid \mathbf{x}_{t-1}^{(i)}, \mathbf{y}_{1:t-1}) w_{t-1}^{(i)}}{q(\mathbf{x}_t^{(i)} \mid \mathbf{y}_{1:t})}$$

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This scheme is called **particle filtering** or *Sequential Importance Sampling* (SIS). Once samples are available, Equation (1) can be used to approximate any integral with respect to $p(\mathbf{x}_t \mid \mathbf{y}_{1:t})$.

Bootstrap filter

If we choose as proposal function

$$q(\mathbf{x}_t \mid \mathbf{y}_{1:t}) = \sum_{i=1}^{N} p(\mathbf{x}_t \mid \mathbf{x}_{t-1}^{(i)}, \mathbf{y}_{1:t-1}) w_{t-1}^{(i)}$$

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$$= p(\mathbf{y}_t \mid \mathbf{x}_t^{(i)})$$

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The resulting algorithm is the **bootstrap filter**, considered the first particle filter

Bootstrap filter: the proposal function

Drawing samples from the proposal

$$q(\mathbf{x}_t \mid \mathbf{y}_{1:t}) = \sum_{i=1}^{N} p(\mathbf{x}_t \mid \mathbf{x}_{t-1}^{(i)}, \mathbf{y}_{1:t-1}) w_{t-1}^{(i)}$$

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to get $\mathbf{x}_{t-1}^{(j_1)}, \mathbf{x}_{t-1}^{(j_2)}, \cdots, \mathbf{x}_{t-1}^{(j_t)}$ with $j_1, j_2, \cdots, j_t \in \{1, \cdots, N\}$

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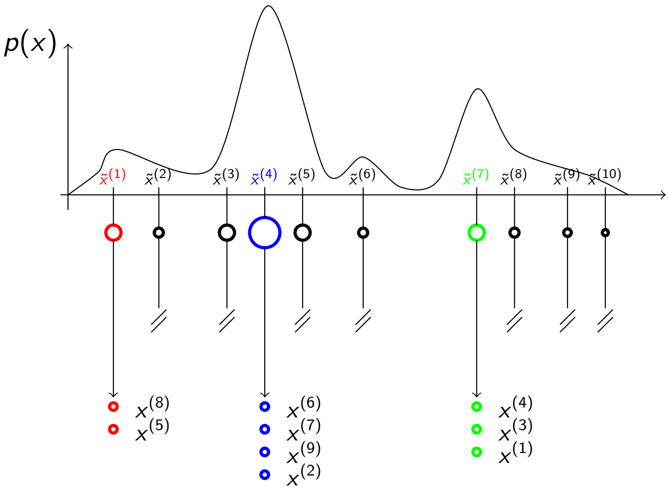
- **propagating** each resampled particle using the transition pdf, $p(\mathbf{x}_t|\mathbf{x}_{t-1})$, as

$$x_t^{(i)} \sim p(\mathbf{x}_t|\mathbf{x}_{t-1}^{(j_i)}), i = 1, \cdots, N$$

Resampling


$p(x)$
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Bootstrap filter: implementation

- **Initialization**
  - sample $\mathbf{x}_0^{(i)}, i = 1, \cdots, N$ from the prior $p(\mathbf{x}_0)$
- **Recursion** given $\hat{p}^N(\mathbf{x}_{t-1} \mid \mathbf{y}_{1:t-1}) = \sum_{i=1}^N w^{(i)} \delta_{\tilde{\mathbf{x}}_{t-1}^{(i)}}$,
  1. resampling: let $\mathbf{x}_{t-1}^{(i)} = \tilde{\mathbf{x}}_{t-1}^{(j)}$ with probability $w^{(j)}, i = 1, \cdots, N, j \in \{1, \cdots, N\}$.
  2. propagation (sampling)

$$\tilde{\mathbf{x}}_t^{(i)} \sim p(\mathbf{x}_t | \mathbf{x}_{t-1}^{(i)}), i = 1, \cdots, N$$

  3. weight computation...

$$w^{*(i)} = p(\mathbf{y}_t \mid \tilde{\mathbf{x}}_t^{(i)}), i = 1, \cdots, N$$

...and normalization

$$w^{(i)} = \frac{w^{*(i)}}{\sum_{j=1}^N w^{*(j)}}, i = 1, \cdots, N$$

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Bootstrap filter: overview

1. **Initialization**

$$\mathbf{x}_0^{(i)} \sim p(\mathbf{x}_0) \text{ for } i = 1, \cdots, N$$

2. **Recursive step:** starting from samples at time instant $t-1$

2.1. **Samples propagation**

$$\tilde{\mathbf{x}}_t^{(i)} \sim p\left(\mathbf{x}_t \mid \mathbf{x}_{t-1}^{(i)}\right)$$

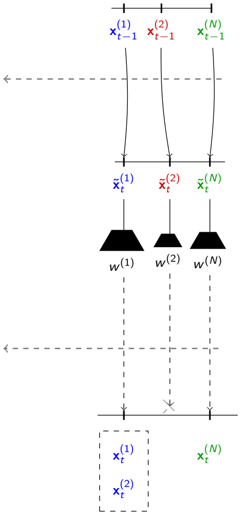
2.2. **Weights computation and normalization**

$$w^{(i)} \propto p\left(\mathbf{y}_t \mid \tilde{\mathbf{x}}_t^{(i)}\right), i = 1, \cdots, N$$

2.3. **Resampling**

$$\mathbf{x}_t^{(i)} = \tilde{\mathbf{x}}_t^{(j)}, i = 1, \cdots, N$$
with probability $w^{(j)}, j \in \{1, \cdots, N\}$

samples at time t



Bootstrap filter: epilogue

In the above implementation, at the end of every iteration we have samples

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but the initial goal was to approximate the expectation of some (known) function of interest, $\mathbf{g}$, with respect to $p(\mathbf{x}_t \mid \mathbf{y}_{1:t})$, i.e.,

$$\mathbb{E}\left[\mathbf{g}(\mathbf{x}_t)\right] = \int \mathbf{g}(\mathbf{x}_t) p(\mathbf{x}_t | \mathbf{y}_{1:t}) d\mathbf{x}_t.$$

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We simply use the samples to compute a Monte Carlo approximation,

$$\mathbb{E}\left[\mathbf{g}(\mathbf{x}_t)\right] \approx \frac{1}{N} \sum_{n=1}^{N} \mathbf{g}(\mathbf{x}_t^{(n)})$$