Channel coding

Introduction & linear codes

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(Channel) Coding

Goal

Add redundancy to the transmitted information so that it can be recovered if errors happen during transmission.

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Example: repetition code

- $0 \to 000$
- $1 \to 111$

so that, e.g.,

$$010 \to 000\,111\,000$$

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Example: repetition code

- $0 \rightarrow 000$
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so that, e.g.,

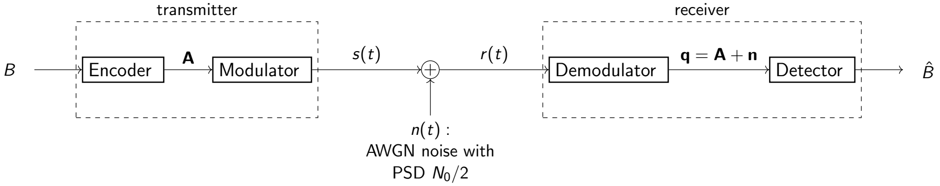
$$010 \rightarrow 000\,111\,000$$

What should we *decide* it was transmitted if we receive

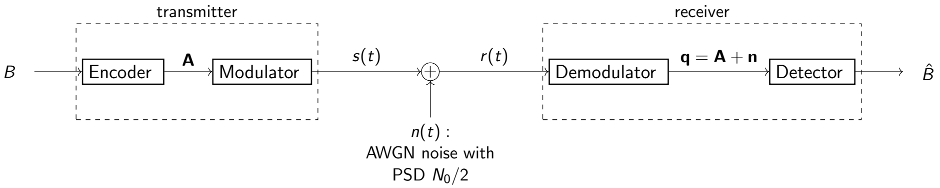
$$010\,100\,000\,?$$

000 (instead of 010)!

Digital communications system



Digital communications system



This model can be analyzed at different levels...

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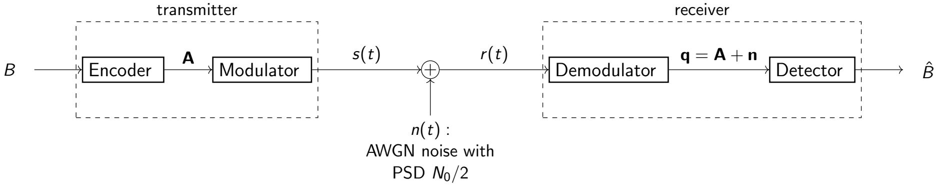

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- Digital channel

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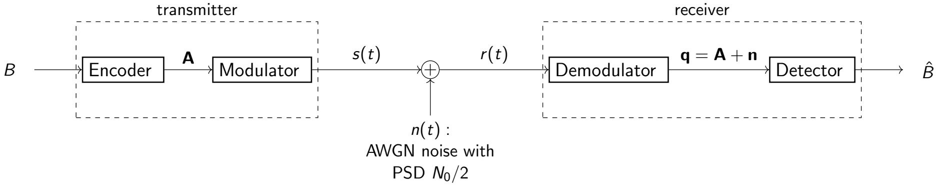
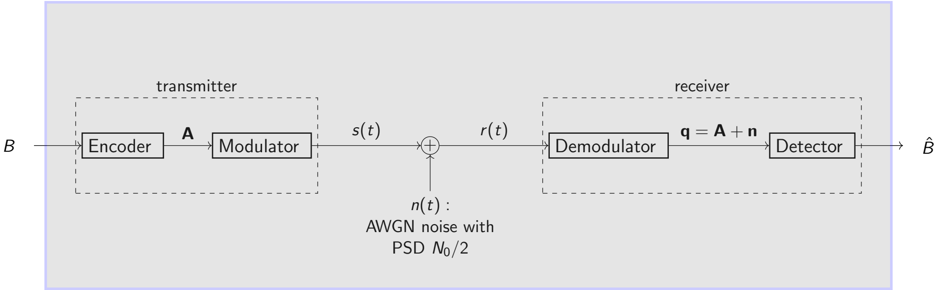


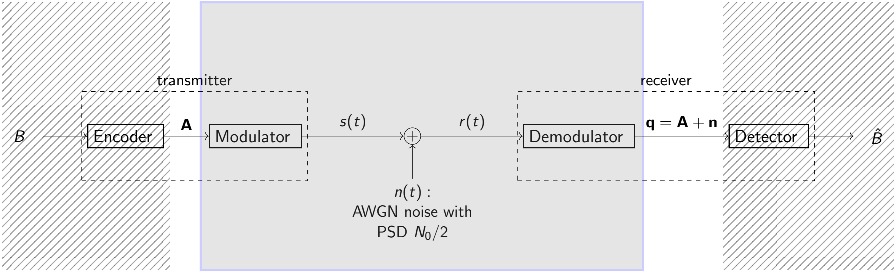
This model can be analyzed at different levels...

- Digital channel
- Gaussian channel

Digital channel

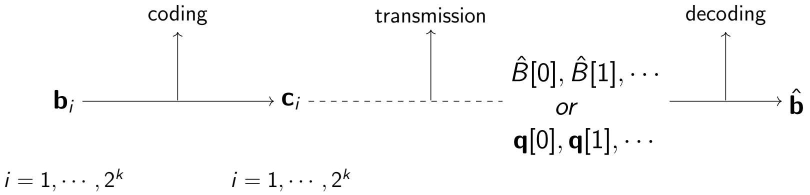


Gaussian channel (with digital input)



Some basic concepts

- **Code**
  Mapping from a sequence of k bits, $\mathbf{b} \in \{\mathbf{b}_1, \mathbf{b}_2, \cdots\}$, onto another one of $n > k$ bits, $\mathbf{c} \in \{\mathbf{c}_1, \mathbf{c}_2, \cdots\}$.



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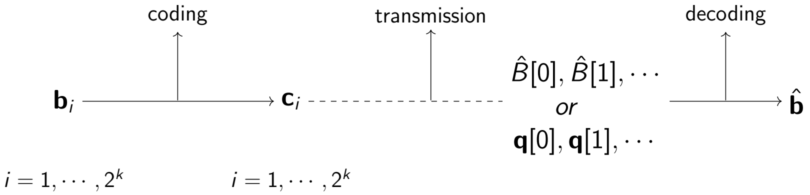

- **Probability of error for $\mathbf{b}_i$**

$$P_e^i = Pr\{\hat{\mathbf{b}} \neq \mathbf{b}_i | \mathbf{b} = \mathbf{b}_i\}, \; i = 1, \ldots, 2^k$$

- **Maximum probability of error**: $P_e^{\max} = \max_i P_e^i$

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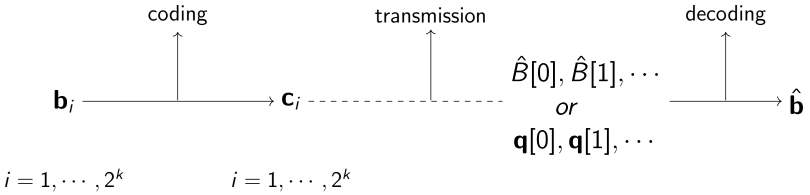

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- **Maximum probability of error**: $P_e^{\max} = \max_i P_e^i$
- **Rate**: The rate of a code is the number of information bits, k, carried by a codeword of length n.

$$R = k/n$$

Codeword vs bit error probability

- P_e: codeword error probability

$$P_e = \frac{\#\text{ codewords received incorrectly}}{\text{overall }\#\text{ codewords}} = \frac{v}{w}$$

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Channel coding theorem

Theorem: Channel coding (Shannon, 1948)

If C is the capacity of a channel, then it is possible to *reliably* transmit with rate $R < C$.

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Capacity

It is the maximum of the mutual information between the input and output of the channel.

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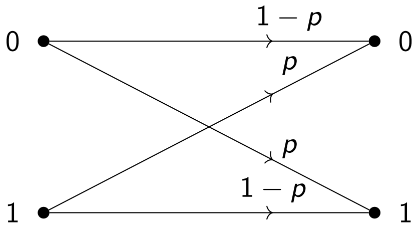
Capacity

It is the maximum of the mutual information between the input and output of the channel.

Reliable transmission

There is a sequence of codes $(n, k) = (n, nR)$ *such that, when* $n \to \infty$, $P_e^{\max} \to 0$.

Channel coding theorem: example



$$C = 1 - H_b(p),$$

being p the channel BER and H_b the binary entropy.

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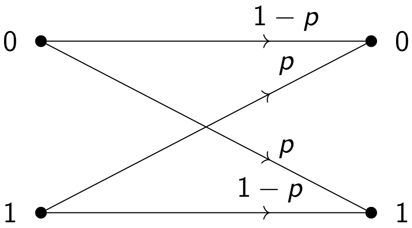

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Let us consider 4 binary channels with

$$p = 0.15 \Rightarrow C_1 = 0.39 \qquad p = 0.13 \Rightarrow C_2 = 0.44$$
$$p = 0.17 \Rightarrow C_3 = 0.34 \qquad p = 0.19 \Rightarrow C_4 = 0.29$$

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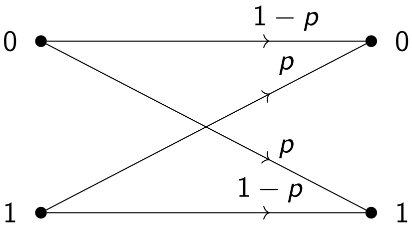

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Channel coding theorem

A code with rate $R = 1/3$ only respects the Shannon limit in the first three scenarios.

Channel coding theorem: example

The figure shows the evolution of the codeword error probability as a function of n: it approaches 0 when $R < C$.

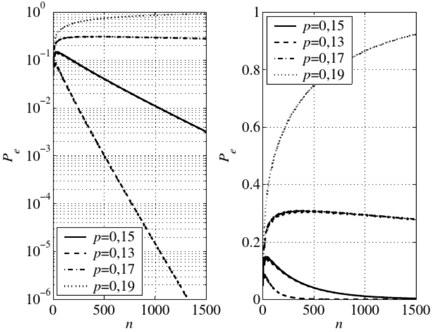

Figure: Left: logarithmic scale; right: linear scale

Definitions

Definition: Redundancy

The number of bits, $r = n - k$, added by the encoder.

It allows rewriting the rate of the code as $R = \frac{k}{n} = \frac{n-r}{n} = 1 - \frac{r}{n}$

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Definition: Hamming distance...

...between two binary sequences is the number of different bits.

It is a measure of how different two sequences of bits are. For instance, $d_H(1010, 1001) = 2$.

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Definition: Minimum distance of a code

$$d_{min} = \min_{i \neq j} d_H(\mathbf{c_i}, \mathbf{c_j})$$

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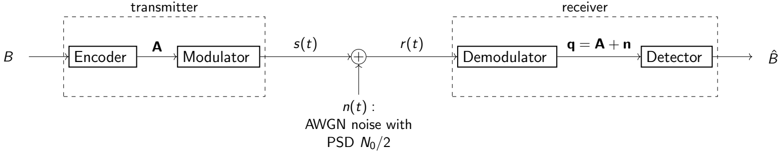
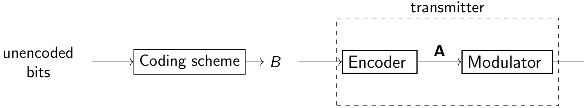
Coding

In the usual model for a digital communications system,



the coding scheme is always placed *before* the system



and we have

$$\left.\begin{aligned} B[0] &= C[0] \\ B[1] &= C[1] \\ \vdots & \quad \vdots \end{aligned}\right\} \text{codeword}$$

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Hard decoding

- Decoding at the *bit level*

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B — Digital channel — $\hat{B}$

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Notation

$$\mathbf{c}_i = \left[C^i[0], C^i[1], \cdots C^i[n-1]\right] \equiv i\text{-th codeword}$$

$$\mathbf{r} = \left[\hat{B}[0], \hat{B}[1], \cdots \hat{B}[n-1]\right] \equiv \text{received word}$$

Hard decoding: decision rule

- Maximum a Posteriori (MAP) rule: we decide $\mathbf{c}_i$ if

$$p(\mathbf{c}_i|\mathbf{r}) > p(\mathbf{c}_j|\mathbf{r}) \qquad \forall j \neq i$$

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$$p(\mathbf{r}|\mathbf{c}_i) = \epsilon^{d_H(\mathbf{r},\mathbf{c}_i)}(1-\epsilon)^{n-d_H(\mathbf{r},\mathbf{c}_i)}$$

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- If $\epsilon < 0.5$ ML rule is tantamount to deciding $\mathbf{c}_i$ if

$$d_H(\mathbf{r},\mathbf{c}_i) < d_H(\mathbf{r},\mathbf{c}_j) \qquad \forall j \neq i.$$

Hard decoding: error detection vs. correction

Assuming errors happened during transmission, there are two possible scenarios:

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We need a *policy* for the latter scenario: in this course we **always** try and fix the errors.

Hard decoding: detection

- We detect a word error when **less than** d_{min} bit errors happen.

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- Probability of an erroneous codeword going **undetected** (at least d_{min} bit errors)

$$P_{nd} \leq \sum_{m=d_{min}}^{n} \binom{n}{m} \epsilon^m (1-\epsilon)^{n-m}$$

where ϵ is the bit error probability in the system, and d_{min} is the minimum distance between codewords.

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A bound on the probability of error...

...since it might happen that d_{min} bit errors do not turn a codeword into another one $\Rightarrow \leq$ rather than $=$

Hard decoding: correction ("always correct" policy)

- Decoding is correct if there are less than $d_{min}/2$ erroneous bits
  $\Rightarrow$ the code can correct **up to**

$$t = \lfloor (d_{min} - 1)/2 \rfloor \text{ errors.}$$

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Approximate bound

The first element in the summation is a good approximation if ϵ is small and d_{min} large.

Soft decoding

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Notation

$$m \equiv \# \text{ bits carried by every } \mathbf{A}$$

$$\tilde{\mathbf{c}}_i = \left[\mathbf{A}^{(i)}[0], \mathbf{A}^{(i)}[1], \cdots \mathbf{A}^{(i)}[n/m-1]\right] \equiv i\text{-th codeword}$$

$$\tilde{\mathbf{r}} = [\mathbf{q}[0], \mathbf{q}[1], \cdots \mathbf{q}[n/m-1]] \equiv \text{received word}$$

Soft decoding: correction

- The *codeword* error probability can be approximated as

$$P_e \approx \kappa \mathrm{Q}\left(\frac{d_{min}/2}{\sqrt{N_0/2}}\right) \quad (1)$$

where κ is the *kiss number*.

Definition: kiss number

It is the maximum number of codewords that are at distance d_{min} from any given.

Coding gain

- If we set equal the *BER* with and without coding, the **coding gain** is obtained as

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To compute the individual E_b/N_0's, it is often useful...

Stirling's approximation

$$Q(x) \approx \frac{1}{2} e^{-\frac{x^2}{2}}$$

Coding gain: example

Let us consider a binary antipodal constellation 2-PAM ($\pm\sqrt{E_s}$), with the code

$\mathbf{b}_i$	$\mathbf{c}_i$
00	000
01	011
10	110
11	101

Coding gain: example - hard decoding

- This code cannot correct any error since $t = \lfloor (d_{min} - 1)/2 \rfloor = 0$, and the codeword error probability is

$$P_e \leq \sum_{m=1}^{3} \binom{3}{m} \epsilon^m (1-\epsilon)^{n-m} \approx 3\epsilon$$

where $\epsilon = Q(\sqrt{2E_s/N_0})$.

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- Bit error probability

$$BER \approx \frac{2}{3} 3Q\left(\sqrt{\frac{2E_s}{N_0}}\right)$$

- In order to express it in terms of E_b, we use that $2E_b = 3E_s$, and hence

$$BER \approx 2Q\left(\sqrt{\frac{4E_b}{3N_0}}\right)$$

Coding gain: example - soft decoding

- We decide **b** from the output of the Gaussian channel,

$$\mathbf{q} = (\mathbf{q}[0], \mathbf{q}[1], \mathbf{q}[2]) = (\mathbf{A}[0] + \mathbf{n}[0], \mathbf{A}[1] + \mathbf{n}[1], \mathbf{A}[2] + \mathbf{n}[2])$$

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- Tantamount to the detector for the constellation

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- BER as a function of E_b:

$$BER \approx 2Q\left(\sqrt{\frac{8E_b}{3N_0}}\right)$$

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- Without coding, we have $E_b = E_s$, and

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- Soft decoding

$$G = 4/3 \approx 1.25dB$$

  - Now we are making good use of coding

Index

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Galois field modulo 2 ($GF(2)$)

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- d_{min} is the smallest weight (number of 1s) among the non-null codewords.

Linear block codes: structure

Elements in an (n, k) linear block code

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Property

Every row of $\mathbf{G}$ is a codeword.

Parity-check matrix

Parity check matrix, $\mathbf{H}$, is the *orthogonal complement* of $\mathbf{G}$ so that

$$\mathbf{c}\mathbf{H}^\top = \mathbf{0} \Leftrightarrow \mathbf{c} \text{ is a codeword}$$

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For the sake of convenience,

Definition: Syndrome

The syndrome of the received sequence $\mathbf{r}$ is

$$\mathbf{s} = \mathbf{r}\mathbf{H}^\top \quad \text{(with dimensions } 1 \times (n-k))$$

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Syndrome-error connection

$$\mathbf{s} = \mathbf{r}\mathbf{H}^T = (\mathbf{c}+\mathbf{e})\mathbf{H}^T = \cancelto{0}{\mathbf{c}\mathbf{H}^T} + \mathbf{e}\mathbf{H}^T = \mathbf{e}\mathbf{H}^T$$

Hard decoding: syndrome decoding

The minimum distance rule requires computing d_H between the received word, $\mathbf{r}$, and every codeword...but we can carry out **syndrome** decoding

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Beforehand:

Fill up a table yielding the syndrome associated with every possible error,

error ($\mathbf{e}$)	syndrome($\mathbf{s}$)
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(If several errors yield the same syndrome, choose the one that is most likely, i.e., the one with the smallest weight)

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3. *Undo* the error

$$\hat{\mathbf{c}} = \mathbf{r} + \mathbf{e}$$

Systematic codes

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- First/last k bits in $\mathbf{c}$ are equal to $\mathbf{b}$, and the remaining $n-k$ are redundancy.
- If $\mathbf{G} = \begin{bmatrix} \mathbf{I}_k & \mathbf{P} \end{bmatrix}$ it can be shown

$$\mathbf{H} = \begin{bmatrix} \mathbf{P}^T & \mathbf{I}_{n-k} \end{bmatrix}$$

Exercise

Prove it!

Systematic code example: Hamming $(7,4)$

generator matrix:

$$\mathbf{G} = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 \end{bmatrix}$$

Parity-check matrix:

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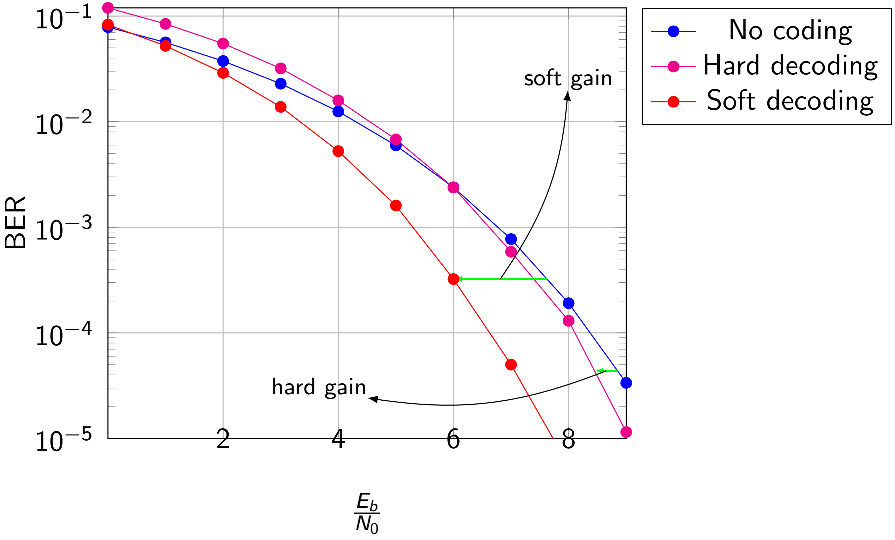
Every Hamming code:

- It's *perfect*
- $d_{min} = 3$
- $k = 2^j - j - 1$ and $n = 2^j - 1 \; \forall j \in \mathbb{N} \geq 2$
  - $j = 2 \rightarrow (3, 1)$
  - $j = 3 \rightarrow (7, 4)$
  - $j = 4 \rightarrow (15, 11)$

Hamming $(7, 4)$: coding gain



Hamming $(7,4)$: decoding

Beforehand we apply

$$\mathbf{s} = \mathbf{e}\mathbf{H}^T$$

over every **e** that entails a single error (the code can only correct 1 erroneous bit):

error	syndrome
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0000100	100
0000010	010
0000001	001

Example: $\mathbf{r} = [1100101]$

$$\mathbf{s} = \mathbf{r}\mathbf{H}^\top = [1100101] \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = [110]$$

and hence $\mathbf{e} = [0100000]$ so that

$$\hat{\mathbf{c}} = \mathbf{r} + \mathbf{e} = \mathbf{r} = [1000101]\,.$$

Equivalent codes

Computing H from G

If the code is systematic, we have an easy way of computing the parity-check matrix...

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Allowed operations are:

- On rows: replace any row with a linear combination of itself and other rows **or** swapping rows.
- On columns: swapping columns.

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Definition: Equivalent codes

Two codes are equivalent if they have the same codewords (after, maybe, reordering the bits).

Index

Cyclic codes

Large values of k and n

Working with matrices is not efficient!!

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Working with matrices is not efficient!!

Definition: Cyclic code

It is a linear block code in which any *circular* shift of a codeword results in another codeword.

In a cyclic code,

- If $[c_0, c_1, \ldots, c_{n-1}]$ is a codeword, then so is $[c_{n-1}, c_0, c_1, \ldots, c_{n-2}]$
  - i.e., every codeword is a (circularly) shifted version of another codeword.

Polynomial representation of codewords

Codeword $[c_0, c_1, \cdots, c_{n-1}]$ is represented as the polynomial

$$c(x) = c_0 + c_1 x + c_2 x^2 + \cdots + c_{n-1} x^{n-1}$$

How is

$$[c_0, c_1, \cdots, c_{n-1}] \rightarrow [c_{n-1}, c_0, \cdots, c_{n-2}]$$

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achieved mathematically? By multiplying $c(x)$ times x modulo $(x^n - 1)$, i.e.,

$$\begin{aligned} xc(x) &= c_0 x + c_1 x^2 + \cdots + c_{n-1} x^n = c_0 x + \cdots + c_{n-1} x^n + c_{n-1} - c_{n-1} \\ &= c_{n-1}(x^n - 1) + c_{n-1} + c_0 x + c_1 x^2 + \cdots + c_{n-2} x^{n-1} \end{aligned}$$

Hence,

$$(xc(x))_{x^n-1} = \underbrace{c_{n-1} + c_0 x + c_1 x^2 + \cdots + c_{n-2} x^{n-1}}_{[c_{n-1}, c_0, \cdots, c_{n-2}]}$$

Encoding

$\mathbf{G}$ generator matrix	$\rightarrow$	$g(x)$ generator polynomial

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Coding is carried out by multiplying, modulo $x^n - 1$, the polynomial representing $\mathbf{b}_i$ by a **generator polynomial**, $g(x)$,

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The generator polynomial, $g(x)$,

- it is of degree $r = n - k$,
- it must be an irreducible polynomial

Decoding

$\mathbf{H}$	$\rightarrow$	$h(x)$
parity-check matrix		parity-check polynomial

Decoding

H parity-check matrix	$\rightarrow$	$h(x)$ parity-check polynomial

The parity-check polynomial, $h(x)$,

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$$(g(x)h(x))_{x^n-1} = 0.$$

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Just like in regular linear block codes, we can perform **syndrome decoding**,

$$s(x) = (r(x)h(x))_{x^n-1}$$