Channel coding

Introduction & linear codes

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February 20, 2024

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(Channel) Coding

Goal

Add redundancy to the transmitted information so that it can be recovered if errors happen during transmission.

Example: repetition code

- $0 \to 000$
- $1 \to 111$

so that, e.g.,

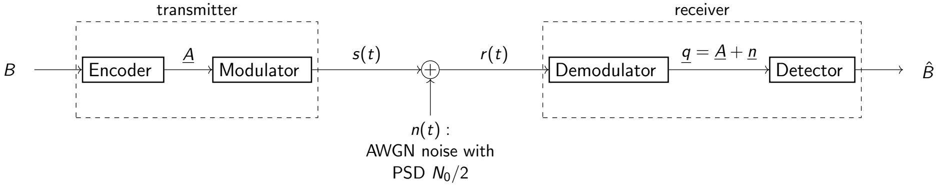
$$010 \to 000\,111\,000$$

What should we *decide* it was transmitted if we receive

$$010\,100\,000\,?$$

000 (instead of 010)!

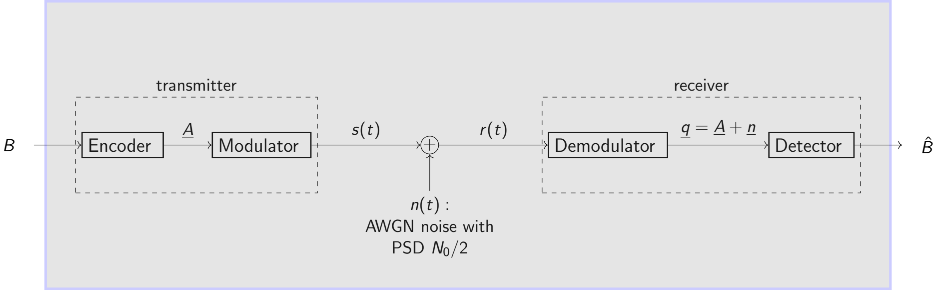
Digital communications system



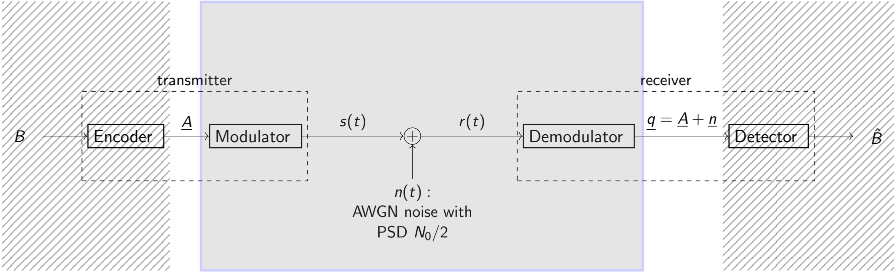
This model can be analyzed at different levels...

- Digital channel
- Gaussian channel

Digital channel

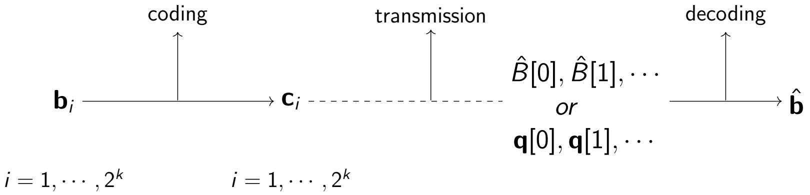


Gaussian channel (with digital input)



Some basic concepts

- **Code**
  Mapping from a sequence of k bits, $\mathbf{b} \in \{\mathbf{b}_1, \mathbf{b}_2, \cdots\}$, onto another one of $n > k$ bits, $\mathbf{c} \in \{\mathbf{c}_1, \mathbf{c}_2, \cdots\}$.



- **Probability of error for $\mathbf{b}_i$**

$$P_e^i = Pr\{\hat{\mathbf{b}} \neq \mathbf{b}_i | \mathbf{b} = \mathbf{b}_i\},\ i = 1, \ldots, 2^k$$

- **Maximum probability of error**: $P_e^{\max} = \max_i P_e^i$
- **Rate**: The rate of a code is the number of information bits, k, carried by a codeword of length n.

$$R = k/n$$

Codeword vs bit error probability

- P_e: codeword error probability

$$P_e = \frac{\#\ \text{codewords received incorrectly}}{\text{overall}\ \#\ \text{codewords}} = \frac{v}{w}$$

- BER (**B**it **E**rror **R**ate): bit error probability

$$BER = \frac{\#\ \text{incorrect bits}}{\#\ \text{transmitted bits}}$$

(they match if every codeword carries a single information bit)

$$\left.\begin{array}{ll} \text{worst-case scenario} \rightarrow & BER = \frac{v \times k}{w \times k} = P_e \\ \text{best-case scenario} \rightarrow & BER = \frac{v \times 1}{w \times k} = \frac{P_e}{k} \end{array}\right\} \Rightarrow \frac{P_e}{k} \leq BER \leq P_e$$

Channel coding theorem

Theorem: Channel coding (Shannon, 1948)

If C is the capacity of a channel, then it is possible to *reliably* transmit with rate $R < C$.

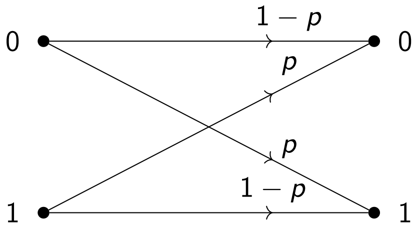
Capacity

It is the maximum of the mutual information between the input and output of the channel.

Reliable transmission

There is a sequence of codes $(n, k) = (n, nR)$ *such that, when* $n \to \infty$, $P_e^{\max} \to 0$.

Channel coding theorem: example



$$C = 1 - H_b(p),$$

being p the channel BER and H_b the binary entropy.

Let us consider 4 binary channels with

$$p = 0.15 \Rightarrow C_1 = 0.39 \qquad p = 0.13 \Rightarrow C_2 = 0.44$$
$$p = 0.17 \Rightarrow C_3 = 0.34 \qquad p = 0.19 \Rightarrow C_4 = 0.29$$

and a code with rate $R = 1/3 = 0.33$.

Channel coding theorem

A code with rate $R = 1/3$ only respects the Shannon limit in the first three scenarios.

Channel coding theorem: example

The figure shows the evolution of the codeword error probability as a function of n: it approaches 0 when $R < C$.

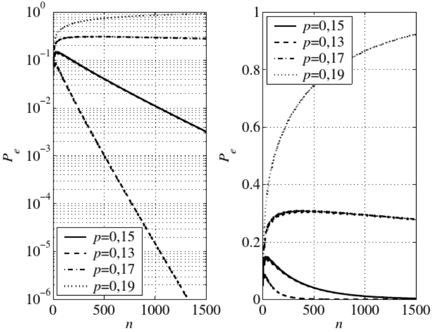


Figure: Left: logarithmic scale; right: linear scale

Definitions

Definition: Redundancy

The number of bits, $r = n - k$, added by the encoder.

It allows rewriting the rate of the code as $R = \frac{k}{n} = \frac{n-r}{n} = 1 - \frac{r}{n}$

Definition: Hamming distance...

...between two binary sequences is the number of different bits.

It is a measure of how different two sequences of bits are. For instance, $d_H(1010, 1001) = 2$.

Definition: Minimum distance of a code

$$d_{min} = \min_{i \neq j} d_H(\mathbf{c_i}, \mathbf{c_j})$$

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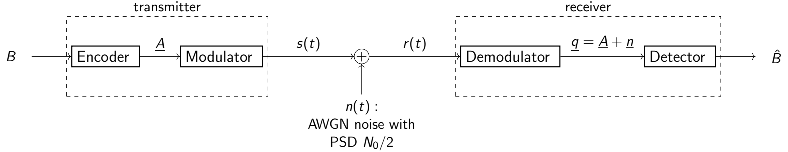
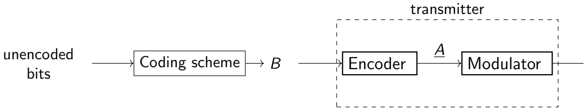
Coding

In the usual model for a digital communications system,



the coding scheme is always placed *before* the system



and we have

$$\left.\begin{aligned} B[0] &= \quad C[0] \\ B[1] &= \quad C[1] \\ \vdots & \qquad\ \vdots \end{aligned}\right\} \text{codeword}$$

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Hard decoding

- Decoding at the *bit level*
- It relies on the digital channel

- The input to the decoder are bits coming from the Detector, the $\hat{B}$'s.
- Metric is the **Hamming distance**.

Notation

$$\mathbf{c}_i = \left[C^i[0], C^i[1], \cdots C^i[n-1]\right] \equiv i\text{-th codeword}$$

$$\mathbf{r} = \left[\hat{B}[0], \hat{B}[1], \cdots \hat{B}[n-1]\right] \equiv \text{received word}$$

Hard decoding: decision rule

- Maximum a Posteriori (MAP) rule: we decide $\mathbf{c}_i$ if

$$p(\mathbf{c}_i|\mathbf{r}) > p(\mathbf{c}_j|\mathbf{r}) \qquad \forall j \neq i$$

- If all the codewords are equally likely, it is equivalent to Maximum Likelihood (ML),

$$p(\mathbf{r}|\mathbf{c}_i) > p(\mathbf{r}|\mathbf{c}_j) \qquad \forall j \neq i$$

  - Likelihoods can be expressed in terms of d_H

$$p(\mathbf{r}|\mathbf{c}_i) = \epsilon^{d_H(\mathbf{r},\mathbf{c}_i)}(1-\epsilon)^{n-d_H(\mathbf{r},\mathbf{c}_i)}$$

$\epsilon \equiv$ *channel* bit error probability

- If $\epsilon < 0.5$ ML rule is tantamount to deciding $\mathbf{c}_i$ if

$$d_H(\mathbf{r},\mathbf{c}_i) < d_H(\mathbf{r},\mathbf{c}_j) \qquad \forall j \neq i.$$

Hard decoding: error detection vs. correction

Assuming errors happened during transmission, there are two possible scenarios:

- We do **not** detect them
  (we only detect errors if $\mathbf{r} \neq \mathbf{c}_i \quad i = 1, \ldots, 2^k$)
- We do detect them, in which case we must make a decision:
  - We don't risk correct them and request a *retransmission*
    (we **cannot** correct *with confidence*)
  - we *try* and correct them
    (a risk is is involved!!)

We need a *policy* for the latter scenario: in this course we **always** try and fix the errors.

Hard decoding: detection

- We detect a word error when **less than** d_{min} bit errors happen.
- Probability of an erroneous codeword going **undetected** (at least d_{min} bit errors)

$$P_{nd} \leq \sum_{m=d_{min}}^{n} \binom{n}{m} \epsilon^{m} (1-\epsilon)^{n-m}$$

where ϵ is the bit error probability in the system, and d_{min} is the minimum distance between codewords.

A bound on the probability of error...

...since it might happen that d_{min} bit errors do not turn a codeword into another one $\Rightarrow \leq$ rather than $=$

Hard decoding: correction ("always correct" policy)

- Decoding is correct if there are less than $d_{min}/2$ erroneous bits $\Rightarrow$ the code can correct **up to**

$$t = \lfloor (d_{min} - 1)/2 \rfloor \text{ errors.}$$

- Error correction probability:

$$P_e \leq \sum_{m=t+1}^{n} \binom{n}{m} \epsilon^m (1-\epsilon)^{n-m}$$

A bound on the probability of error...

...since it is possible to correct more than t errors (there is no guarantee, though) $\Rightarrow \leq$ rather than $=$

Approximate bound

The first element in the summation is a good approximation if ϵ is small and d_{min} large.

Soft decoding

- Decoding at the *element from the constellation level*
- It relies on the Gaussian channel

with

$$\mathbf{q} = \mathbf{A} + \mathbf{n}$$

where $\mathbf{n}$ is a Gaussian noise vector.

- The input to the decoder are the observations coming from the Demodulator, the $\mathbf{q}$'s.
- Metric is **Euclidean distance**

Notation

$m \equiv \#$ bits carried by every $\mathbf{A}$

$$\tilde{\mathbf{c}}_i = \left[\mathbf{A}^{(i)}[0], \mathbf{A}^{(i)}[1], \cdots \mathbf{A}^{(i)}[n/m-1]\right] \equiv i\text{-th codeword}$$

$$\tilde{\mathbf{r}} = [\mathbf{q}[0], \mathbf{q}[1], \cdots \mathbf{q}[n/m-1]] \equiv \text{received word}$$

Soft decoding: correction

- The *codeword* error probability can be approximated as

$$P_e \approx \kappa \mathrm{Q}\left(\frac{d_{min}/2}{\sqrt{N_0/2}}\right) \tag{1}$$

where κ is the *kiss number*.

Definition: kiss number

It is the maximum number of codewords that are at distance d_{min} from any given.

Coding gain

- If we set equal the *BER* with and without coding, the **coding gain** is obtained as

$$G = \frac{(E_b/N_0)_{nc}}{(E_b/N_0)_c}$$

- Different for soft and hard decoding

To compute the individual E_b/N_0's, it is often useful...

Stirling's approximation

$$Q(x) \approx \frac{1}{2} e^{-\frac{x^2}{2}}$$

Coding gain: example

Let us consider a binary antipodal constellation 2-PAM ($\pm\sqrt{E_s}$), with the code

$\mathbf{b}_i$	$\mathbf{c}_i$
00	000
01	011
10	110
11	101

Coding gain: example - hard decoding

- This code cannot correct any error since $t = \lfloor (d_{min} - 1)/2 \rfloor = 0$, and the codeword error probability is

$$P_e \leq \sum_{m=1}^{3} \binom{3}{m} \epsilon^m (1-\epsilon)^{n-m} \approx 3\epsilon$$

where $\epsilon = Q(\sqrt{2E_s/N_0})$.

- Bit error probability

$$BER \approx \frac{2}{3} 3Q\left(\sqrt{\frac{2E_s}{N_0}}\right)$$

- In order to express it in terms of E_b, we use that $2E_b = 3E_s$, and hence

$$BER \approx 2Q\left(\sqrt{\frac{4E_b}{3N_0}}\right)$$

Coding gain: example - soft decoding

- We decide **b** from the output of the Gaussian channel,

$$\mathbf{q} = (\mathbf{q}[0], \mathbf{q}[1], \mathbf{q}[2]) = (\mathbf{A}[0] + \mathbf{n}[0], \mathbf{A}[1] + \mathbf{n}[1], \mathbf{A}[2] + \mathbf{n}[2])$$

- Tantamount to the detector for the constellation

$$\begin{pmatrix} -\sqrt{E_s} \\ -\sqrt{E_s} \\ -\sqrt{E_s} \end{pmatrix}, \quad \begin{pmatrix} -\sqrt{E_s} \\ \sqrt{E_s} \\ \sqrt{E_s} \end{pmatrix}, \quad \begin{pmatrix} \sqrt{E_s} \\ \sqrt{E_s} \\ -\sqrt{E_s} \end{pmatrix}, \quad \begin{pmatrix} \sqrt{E_s} \\ -\sqrt{E_s} \\ \sqrt{E_s} \end{pmatrix}$$

  which has minimum (Euclidean) distance $d_{min} = 2\sqrt{2E_s}$
- From (1) the codeword error probability is

$$P_e \approx 3Q\left(\sqrt{\frac{4E_s}{N_0}}\right)$$

- BER as a function of E_b:

$$BER \approx 2Q\left(\sqrt{\frac{8E_b}{3N_0}}\right)$$

Coding gain: example - hard vs soft decoding

- Without coding, we have $E_b = E_s$, and

$$\mathrm{BER}_{nc} = \epsilon = Q\left(\sqrt{2E_b/N_0}\right)$$

- Gain with hard decoding
  - We set equal BER_c and BER_{nc}
  - Approximation: $Q(\cdot)$

$$G = \frac{(E_b/N_0)_{nc}}{(E_b/N_0)_c} = 2/3 \approx -1.76dB$$

  - We are actually losing performance!! (expected, since the code is not able correct any error)
- Soft decoding

$$G = 4/3 \approx 1.25dB$$

  - Now we are making good use of coding

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Linear block codes

Galois field modulo 2 ($GF(2)$)

$$a + b = (a + b)_2$$
$$a \cdot b = (a \cdot b)_2$$

Definition: Linear Block Code

A linear block code is a code in which any linear combination of codewords is also a codeword.

Properties

- It is a subspace in $GF(2)^n$ with 2^k elements.
- The all-zeros word is a codeword.
- Every codeword has at least another codeword that is at d_{min} from it.
- d_{min} is the smallest weight (number of 1s) among the non-null codewords.

Linear block codes: structure

Elements in an (n, k) linear block code

- **b** is the message, $1 \times k$
- **c** is the codeword, $1 \times n$
- **r** is the received word, $1 \times n$ with

$$\mathbf{r} = \mathbf{c} + \mathbf{e}$$

  - **e** is the noise $1 \times n$
- **G** is the **generator** matrix,

  (for encoding) $k \times n$

- **H** is the **parity-check** matrix,

  (for decoding) $n - k \times n$

Encoding

The mapping $\mathbf{b} \rightarrow \mathbf{c}$ is performed through matrix multiplication i.e.,

$$\mathbf{c} = \mathbf{bG}.$$

Keep in mind:

- $\mathbf{b}$ is $1 \times k$
- $\mathbf{G}$ is $k \times n$
- $\mathbf{c}$ is $1 \times n$

Property

Every row of $\mathbf{G}$ is a codeword.

Parity-check matrix

Parity check matrix, $\mathbf{H}$, is the *orthogonal complement* of $\mathbf{G}$ so that

$$\mathbf{c}\mathbf{H}^\top = \mathbf{0} \Leftrightarrow \mathbf{c} \text{ is a codeword}$$

For the sake of convenience,

Definition: Syndrome

The syndrome of the received sequence $\mathbf{r}$ is

$$\mathbf{s} = \mathbf{r}\mathbf{H}^\top \quad \text{(with dimensions } 1 \times (n-k))$$

Then,

$$\mathbf{s} = \mathbf{0} \Leftrightarrow \mathbf{r} \text{ is a codeword.}$$

Syndrome-error connection

$$\mathbf{s} = \mathbf{r}\mathbf{H}^T = (\mathbf{c}+\mathbf{e})\mathbf{H}^T = \cancelto{0}{\mathbf{c}\mathbf{H}^T} + \mathbf{e}\mathbf{H}^T = \mathbf{e}\mathbf{H}^T$$

Hard decoding: syndrome decoding

The minimum distance rule requires computing d_H between the received word, $\mathbf{r}$, and every codeword...but we can carry out **syndrome** decoding

Beforehand:

Fill up a table yielding the syndrome associated with every possible error,

error ($\mathbf{e}$)	syndrome($\mathbf{s}$)
⋮	⋮

(If several errors yield the same syndrome, choose the one that is most likely, i.e., the one with the smallest weight)

In operation: given the received word, $\mathbf{r}$:

1. Compute the syndrome $\mathbf{s} = \mathbf{r}\mathbf{H}^T$.
2. Look up the table for the error pattern, $\mathbf{e}$, with that syndrome
3. *Undo* the error

$$\hat{\mathbf{c}} = \mathbf{r} + \mathbf{e}$$

Systematic codes

Definition: Systematic code

A code in which the message is always embedded in the encoded sequence (in the same place).

This can be easily imposed through the generator matrix,

$$\mathbf{G} = \begin{bmatrix} \mathbf{I}_k & \mathbf{P} \end{bmatrix} \quad \text{or} \quad \mathbf{G} = \begin{bmatrix} \mathbf{P} & \mathbf{I}_k \end{bmatrix}$$

- First/last k bits in $\mathbf{c}$ are equal to $\mathbf{b}$, and the remaining $n-k$ are redundancy.
- If $\mathbf{G} = \begin{bmatrix} \mathbf{I}_k & \mathbf{P} \end{bmatrix}$ it can be shown

$$\mathbf{H} = \begin{bmatrix} \mathbf{P}^T & \mathbf{I}_{n-k} \end{bmatrix}$$

Exercise

Prove it!

Systematic code example: Hamming $(7,4)$

generator matrix:

$$\mathbf{G} = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 \end{bmatrix}$$

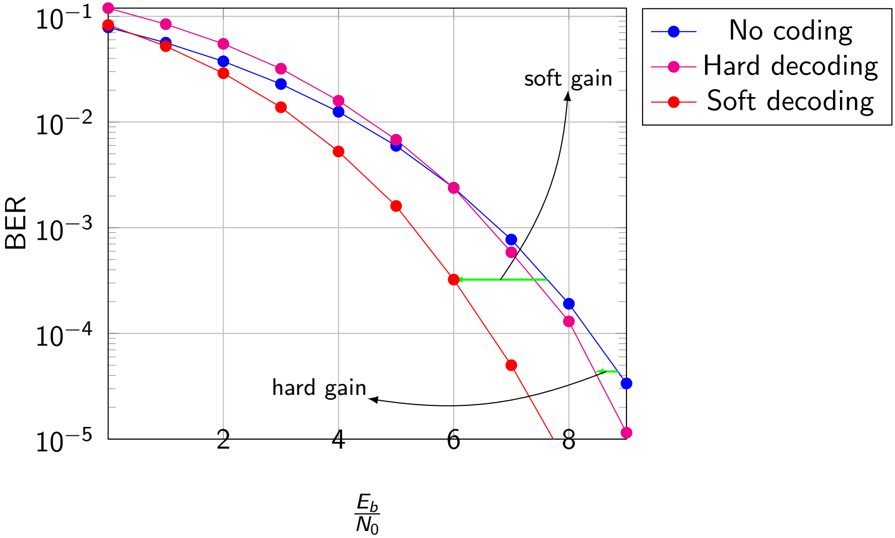
Parity-check matrix:

$$\mathbf{H} = \begin{bmatrix} 1 & 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 & 0 & 0 & 1 \end{bmatrix}$$

Every Hamming code:

- It's *perfect*
- $d_{min} = 3$
- $k = 2^j - j - 1$ and $n = 2^j - 1 \ \forall j \in \mathbb{N} \geq 2$
  - $j = 2 \rightarrow (3,1)$
  - $j = 3 \rightarrow (7,4)$
  - $j = 4 \rightarrow (15,11)$

Hamming (7, 4): coding gain



Hamming $(7,4)$: decoding

Beforehand we apply

$$\mathbf{s} = \mathbf{e}\mathbf{H}^T$$

over every $\mathbf{e}$ that entails a single error (the code can only correct 1 erroneous bit):

error	syndrome
0000000	000
1000000	101
0100000	110
0010000	111
0001000	011
0000100	100
0000010	010
0000001	001

Example: $\mathbf{r} = [1100101]$

$$\mathbf{s} = \mathbf{r}\mathbf{H}^\top = [1100101] \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = [110]$$

and hence $\mathbf{e} = [0100000]$ so that

$$\hat{\mathbf{c}} = \mathbf{r} + \mathbf{e} = \mathbf{r} = [1000101]\,.$$

Equivalent codes

Computing H from G

If the code is systematic, we have an easy way of computing the parity-check matrix...

...but what if it's not? If the code is **not** systematic, one can apply operations on the generator matrix, $\mathbf{G}$, to try and transform it into that of an *equivalent* systematic code, $\mathbf{G}' = \begin{bmatrix} \mathbf{I}_k & \mathbf{P} \end{bmatrix}$.

Allowed operations are:

- **On rows** replace any row with a linear combination of itself and other rows **or** swapping rows.
- **On columns** swapping columns.

Definition: Equivalent codes

Two codes are equivalent if they have the same codewords (after, maybe, reordering the bits).

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Cyclic codes

Large values of k and n

Working with matrices is not efficient!!

Definition: Cyclic code

It is a linear block code in which any *circular* shift of a codeword results in another codeword.

In a cyclic code,

- If $[c_0, c_1, \ldots, c_{n-1}]$ is a codeword, then so is $[c_{n-1}, c_0, c_1, \ldots, c_{n-2}]$
  - i.e., every codeword is a (circularly) shifted version of another codeword.

Polynomial representation of codewords

Codeword $[c_0, c_1, \cdots, c_{n-1}]$ is represented as the polynomial

$$c(x) = c_0 + c_1 x + c_2 x^2 + \cdots + c_{n-1} x^{n-1}$$

How is

$$[c_0, c_1, \cdots, c_{n-1}] \to [c_{n-1}, c_0, \cdots, c_{n-2}]$$

achieved mathematically? By multiplying $c(x)$ times x modulo $(x^n - 1)$, i.e.,

$$\begin{aligned} xc(x) &= c_0 x + c_1 x^2 + \cdots + c_{n-1} x^n = c_0 x + \cdots + c_{n-1} x^n + c_{n-1} - c_{n-1} \\ &= c_{n-1}(x^n - 1) + c_{n-1} + c_0 x + c_1 x^2 + \cdots + c_{n-2} x^{n-1} \end{aligned}$$

Hence,

$$(xc(x))_{x^n-1} = \underbrace{c_{n-1} + c_0 x + c_1 x^2 + \cdots + c_{n-2} x^{n-1}}_{[c_{n-1}, c_0, \cdots, c_{n-2}]}$$

Encoding

$\mathbf{G}$ generator matrix	$\rightarrow$	$g(x)$ generator polynomial

Coding is carried out by multiplying, modulo $x^n - 1$, the polynomial representing $\mathbf{b}_i$ by a **generator polynomial**, $g(x)$,

$$c(x) = (b(x)g(x))_{x^n-1}$$

The generator polynomial, $g(x)$,

- it is of degree $r = n - k$,
- it must be an irreducible polynomial

Decoding

$\mathbf{H}$ parity-check matrix	$\rightarrow$	$h(x)$ parity-check polynomial

The parity-check polynomial, $h(x)$,

- it is of degree $r' = n - k - 1$,
- must satisfy

$$(g(x)h(x))_{x^n-1} = 0.$$

Just like in regular linear block codes, we can perform **syndrome decoding**,

$$s(x) = (r(x)h(x))_{x^n-1}$$